Combinatoire et dynamique du flot de Teichmüller

Vincent Delecroix

Université de la Méditerranée
Faculté des sciences de Luminy

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présentée par **Vincent Delecroix**

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Directeur de thèse : Arnaldo Nogueira

Rapporteurs :

William Veech, *Rice University, Houston*
Jean-Cristophe Yoccoz, *Collège de France, Paris*

Jury :

Pierre Arnoux, *Université de la Méditerranée, Marseille*
Pascal Hubert, *Université Paul Cézanne, Marseille*
Arnaldo Nogueira, *Université de la Méditerranée, Marseille*
Jean-Cristophe Yoccoz, *Collège de France, Paris*
Anton Zorich, *Université de Rennes 1*

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Introduction

Ce travail de thèse est une étude dynamique de *billards rationnels* : une particule se déplace dans un polygone dont les angles sont rationnels et rebondit de manière élastique sur ses arêtes. Les *surfaces de translation* ont été introduites pour étudier les billards rationnels et les généralisent. W. Veech (1982) et H. Masur (1982) conçoivent l'ensemble des surfaces de translation comme un espace géométrique sur lequel opère un flot de *renormalisation* (le *flot de Teichmüller*) permettant d'étudier la dynamique à long terme d'une surface de translation. Les versions combinatoires de cette renormalisation, en particulier les inductions de Rauzy (1979) et de Ferenczi-Zamboni (2010), offrent une version concrète de ce flot. La plupart des résultats sur les billards rationnels utilisent le flot de Teichmüller dans sa version géométrique ou combinatoire.

Dans un premier temps, nous nous intéressons à une famille de billards infinis appelée le *vent dans les arbres* introduite par P. et T. Ehrenfest (1912). La version périodique de J. Hardy et J. Weber (1980) qui nous intéresse est un plan (infini) dans lequel des obstacles rectangulaires identiques sont disposés périodiquement. D'une part, nous construisons des familles de *trajectoires divergentes* et donnons ainsi une contrepartie à un théorème de P. Hubert, S. Lelièvre et S. Troubetzkoy. Cette construction repose sur l'algorithme de renormalisation de Ferenczi-Zamboni. D'autre part, nous démontrons que le *taux de diffusion* dans ce billard est génériquement $2/3$: la distance maximale atteinte avant le temps t par une particule se déplaçant à vitesse 1 est de l'ordre de $t^{2/3}$. Ce travail s'appuie sur la renormalisation par le flot de Teichmüller et une généralisation des travaux d'A. Zorich (1996,1997,1999) et G. Forni (2002) sur les déviations des sommes de Birkhoff dans les surfaces de translation.

L'autre type de résultats que nous présentons est de nature combinatoire. Nous étudions la version discrète du flot de Teichmüller donnée par l'induction de Rauzy. Cet algorithme utilise des graphes dont les sommets sont des *permutations irréductibles*. L'ensemble des sommets d'un graphe est une *classe de Rauzy*. En utilisant une interprétation géométrique de ces permutations en terme de surfaces de translation, nous établissons une formule explicite donnant le nombre de permutations dans chaque classe de Rauzy.

Cette thèse commence par une introduction à la dynamique des surfaces de translation et leur renormalisation par le flot de Teichmüller, le coeur de notre travail se situant dans trois articles indépendants en annexe.

Bibliographie thématique

Cette courte bibliographie thématique donne des pistes aux lecteurs et situe notre travail. Les références complètes sont données à la fin.

Échanges d'intervalles et surfaces de translation

Trois survols introduisent les échanges d'intervalles et les surfaces de translation :

[MT02] H. Masur, S. Tabachnikov *Rational billiards and flat structures.*

[Zor06] A. Zorich, *Flat surfaces.*

[Via] M. Viana, *Dynamics of interval exchange maps and Teichmüller flows.*

On pourra également se reporter aux articles fondateurs [KZ75], [Kea75], [Rau79], [Mas82], [Vee82], [KMS86], [Vee89], etc. Pour l'induction de Rauzy on peut consulter l'article original [Rau79] ou les trois ouvrages ci-dessus. Trois articles traitent de l'induction de Ferenczi-Zamboni :

[FZ10] S. Ferenczi, L. Zamboni, *Structure of k-interval exchange transformations : induction, trajectories, and distance theorems.*

[FZ11] S. Ferenczi, L. Zamboni, *Eigenvalues and simplicity of 4 interval exchange transformations.*

[CFZ] J. Cassaigne, S. Ferenczi, L. Zamboni, *Combinatorial trees arising in the study of interval exchange transformations.*

Le dernier de ces articles contient une formule pour la cardinalité de certains graphes intervenant dans cette induction. L'article [Dela] traite du même problème pour l'induction de Rauzy.

Déviations des moyennes ergodiques pour le flot linéaire des surfaces de translation

Sur le sujet plus spécifique des déviations de sommes de Birkhoff (ou taux de diffusion), les trois articles d'A. Zorich utilisent la version combinatoire de la renormalisation tandis que celui de Forni développe une approche géométrique.

[Zor96] A. Zorich, *Finite Gauss measure on the space of interval exchange transformations, Lyapunov exponents.*

[Zor97] A. Zorich, *Deviation for interval exchange transformations.*

[Zor99] A. Zorich, *How do the leaves of a closed 1-form wind around a surface.*

[For02] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus.*

Dans notre article [DHL], nous généralisons une partie des résultats ci-dessus et l'appliquons au vent dans les arbres.

Dynamique du vent dans les arbres

Les deux articles originaux n'utilisent pas les surfaces de translation, tandis qu'elles sont centrales dans le troisième.

[EE12] P. et T. Ehrenfest, *The conceptual foundations of the statistical approach in mechanics.*

[HW80] J. ,Hardy, J. Weber, *Diffusion in a periodic wind-tree model.*

[HLT], P. Hubert, S. Lelièvre, S. Troubetzkoy, *The Ehrenfest wind-tree model : periodic directions, recurrence, diffusion.*

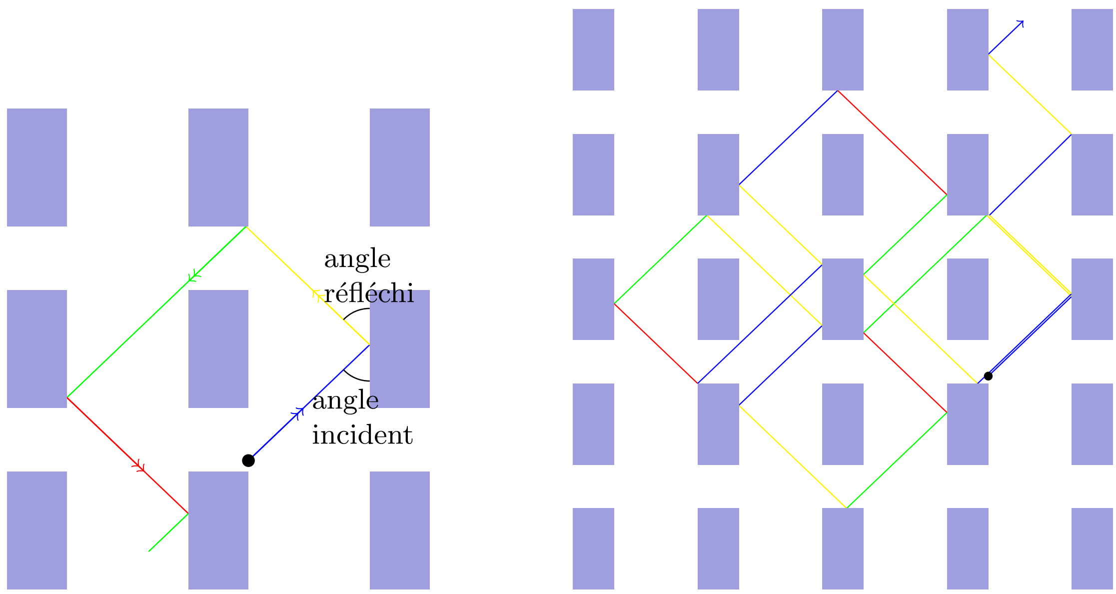
Nos articles [Delb] et [DHL] prolongent ces études.

Table des matières

Chapitre 1

Le vent dans les arbres

Nous décrivons dans cette partie une famille de billards appelée le *vent dans les arbres*. L'origine de ce modèle (et son nom traduit de l'anglais *wind-tree*) remonte à P. et T. Ehrenfest [EE12] dans une étude des lois de Boltzmann de la thermodynamique. Leur modèle est une version du gaz de Lorentz dans lequel les obstacles sont des rectangles et non plus des disques (on trouve parfois le nom de *gaz de Lorentz rectangulaire*). En 1980, J. Hardy et J. Weber [HW80] ont introduit le modèle périodique du vent dans les arbres qui nous intéresse ici. Ce dernier est construit de la manière suivante. Considérons le plan $\mathbb{R}^2$ dans lequel sont placés des obstacles rectangulaires identiques disposés à chaque point de coordonnées entières. Les côtés des rectangles sont supposés horizontaux et verticaux et on appelle respectivement a et b leurs longueurs. Notons $V(a,b)$ le plan $\mathbb{R}^2$ privé de ces rectangles. Une particule (identifiée à un point) se déplace dans $V(a,b)$ en ligne droite (le *vent*) et rebondit sur les rectangles (les *arbres*) selon la loi de l'optique géométrique : l'angle incident est égal à l'angle réfléchi (voir la figure 1.1).



(a) Les rebonds sur les obstacles rectangles du billard infini $V(a,b)$.

(b) Un morceau de trajectoire dans $V(a,b)$ qui prolonge celle de la figure 1.1a.

FIGURE 1.1 – Un morceau de trajectoire dans la table de billard $V(a,b)$ avec $a = 0.33$ et $b = 0.65$.

1.1 Le flot du vent dans les arbres

Nous fixons des paramètres a et b entre 0 et 1 pour toute cette section.

Considérons une particule dans la table de billard $V(a,b)$ et notons $\theta \in S^1$ l'angle de son vecteur vitesse mesuré par rapport à l'horizontale au temps $t = 0$. Comme les rebonds de la particule sur les obstacles se font selon des côtés horizontaux et verticaux, les angles successifs du vecteur vitesse de la particule prennent quatre valeurs distinctes : θ, $-\theta$, $\pi - \theta$ et $\pi + \theta$. Soient $h : \sigma \mapsto -\sigma$ et $v : \sigma \mapsto \pi - \sigma$ les réflexions horizontale et verticale respectivement, et soit $K = \{1, h, v, hv\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ le groupe qu'elles engendrent (voir la figure 1.2).

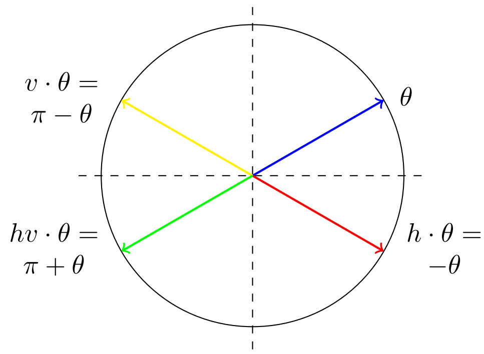


FIGURE 1.2 – Étant donnée une direction initiale θ, une particule du vent dans les arbres prend au plus quatre directions.

Fixons un angle θ. Une particule se déplaçant dans le billard $V(a,b)$ avec direction initiale θ est déterminé par deux paramètres :

- sa *position* : un élément de $V(a,b)$;
- sa *direction* relativement à θ : un élément de K.

On appellera un couple position-direction un *état*. Le *flot dans la direction* θ du billard $V(a,b)$ est la fonction $\phi^\theta : \mathbb{R} \times V(a,b) \times K \to V(a,b) \times K$ qui à la donnée d'un temps $T \in \mathbb{R}$ et de l'état initial d'une particule $(p,g) \in V(a,b) \times K$ associe son état au temps T. Nous utiliserons la notation $\phi^\theta_T(p,g) = \phi^\theta(T,p,g)$. Comme la loi de déplacement de la particule ne dépend pas du temps, nous avons $\phi^\theta_{T+T'} = \phi^\theta_T \circ \phi^\theta_{T'}$.

Nous menons une étude de la dynamique du billard $V(a,b)$ en essayant de suivre le schéma général suivant :

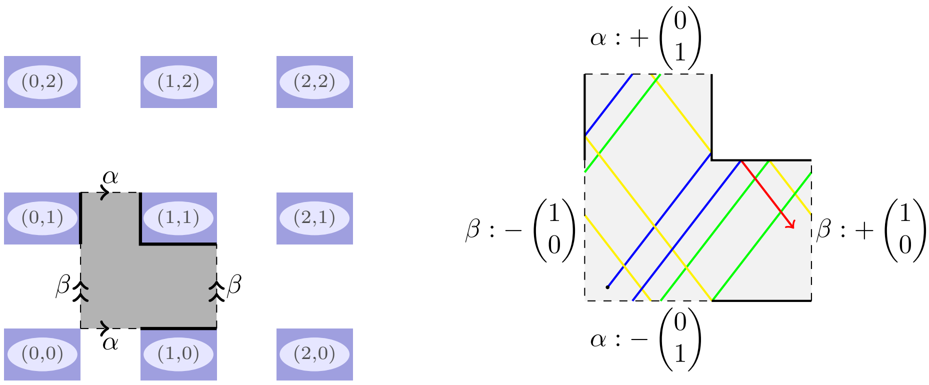
- « Décrire » les comportements possibles des trajectoires.
- Étant donnée la taille des obstacles rectangulaires $a \times b$, existe t-il une « trajectoire typique » dans le billard $V(a,b)$ (relativement à l'angle θ et au point de départ p) ?
- Quantifier la taille de l'ensemble des paramètres $(p,\theta) \in V(a,b) \times S^1$ dont la trajectoire associée n'a pas un comportement typique.
- Comment ces résultats dépendent des paramètres a et b ?

Comme la table de billard $V(a,b)$ est $\mathbb{Z}^2$ périodique, son flot ϕ^θ_T l'est aussi. Nous allons utiliser cette périodicité pour donner une autre description de ϕ^θ_T.

1.2 Le billard quotient et le cocycle du vent dans les arbres

Le billard quotient $\overline{V}(a,b) = V(a,b)/\mathbb{Z}^2$ est le quotient de la table de billard par les translations de $\mathbb{Z}^2$. Le flot ϕ_T^θ du vent dans les arbres commute avec l'action de $\mathbb{Z}^2$ et donne un flot $\overline{\phi}_T^\theta$ dans le quotient. Dans cette section, nous construisons un billard pour décrire $\overline{V}(a,b)$ et définissons le *cocycle du vent dans les arbres* qui fait le lien entre la dynamique dans $\overline{V}(a,b)$ et $V(a,b)$.

Considérons le domaine fondamental de l'action de $\mathbb{Z}^2$ sur $V(a,b)$ donné par le carré dont le coin inférieur gauche est $(a/2, b/2)$ et supérieur droit $(a/2+1, b/2+1)$ (voir figure 1.3a). Le quotient de la table de billard $V(a,b)$ par $\mathbb{Z}^2$ s'identifie naturellement à ce domaine fondamental dans lequel on a recollé les côtés dénotés α et β dans la figure 1.3a. Le flot ϕ_T^θ du billard passe aussi au quotient et on obtient un billard dans un domaine d'aire finie (qui a une forme de L sur la figure 1.3b).



(a) Domaine fondamental pour l'action de $\mathbb{Z}^2$ sur $V(a,b)$. Les arêtes α et β sont identifiées dans le billard quotient (voir figure 1.3b).

(b) Billard dans le quotient $\overline{V}(a,b)$. Il faut identifier les deux arêtes horizontales en ligne brisée dénotés α ainsi que les arêtes β. Les vecteurs indiquent le déplacement dans la table de billard $V(a,b)$.

FIGURE 1.3 – Domaine fondamental et billard quotient du vent dans les arbres $V(a,b)$.

Étant donné une trajectoire dans le billard quotient $\overline{V}(a,b)$, il est facile de retrouver la trajectoire du vent dans les arbres qui lui correspond : à chaque fois que la trajectoire traverse le côté dénoté $\beta : +\begin{pmatrix}1\\0\end{pmatrix}$ dans la figure 1.3a, on se déplace d'un cran vers la droite. Autrement dit on ajoute le vecteur $\begin{pmatrix}1\\0\end{pmatrix}$ à la position courante. On fait de même pour les autres côtés. Ainsi, une manière équivalente de voir le flot ϕ_T^θ du billard $V(a,b)$ est de considérer :

- un élément de $\mathbb{Z}^2$ qui correspond à la copie du domaine fondamental que la particule est en train de visiter (voir la figure 1.3b) ;
- un état position-direction du billard quotient : un élément de $\overline{V}(a,b) \times K$ (voir la figure 1.4).

Le *cocycle du vent dans les arbres* est la procédure qui à une trajectoire dans le billard quotient décrit les déplacements dans $\mathbb{Z}^2$ à effectuer pour retrouver une trajectoire du vent dans les arbres. Formellement, il s'agit de la suite de fonctions $f^{(T)} : \overline{V}(a,b) \times K \to \mathbb{Z}^2$, qui à un état $(p,g) \in \overline{V}(a,b)$ associe le vecteur de $\mathbb{Z}^2$ correspondant au déplacement de domaine fondamental dans $V(a,b)$ effectuée par une particule partant d'un relèvement de (p,g) pendant un temps T. Cette fonction $f^{(T)}$ est discontinue en T : elle fait des sauts chaque fois que la particule passe d'un domaine fondamental à un autre. En utilisant une application de Poincaré, nous construisons dans la section 1.4, une version discrète de $\overline{\phi}_T^\theta$ donné par un *échange d'intervalles* et du cocycle $f^{(T)}$ qui devient une *somme de Birkhoff*.

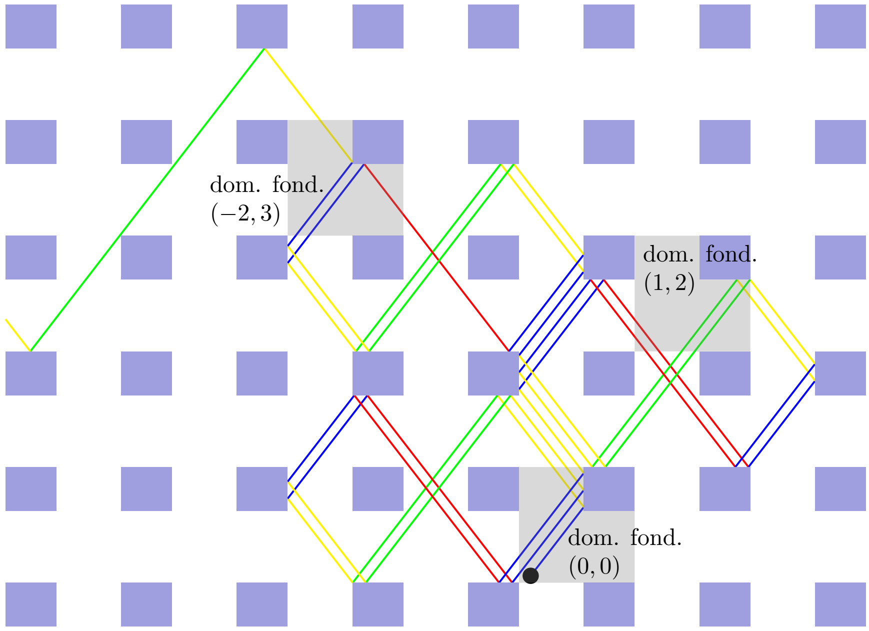


FIGURE 1.4 – La trajectoire de la figure 1.3b relevée dans la table de billard $V(a,b)$ (la trajectoire est ici beaucoup plus longue). Trois translatés du domaine fondamental sont dessinés en gris.

1.3 Passage du billard quotient à la surface de translation $\mathrm{X}(a,b)$

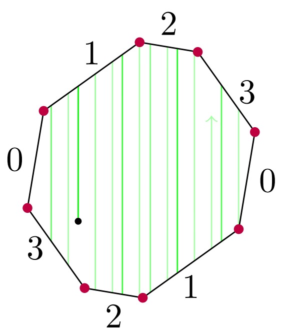


FIGURE 1.5 – Une surface de translation et une orbite de son flot linéaire. Les recollements des arêtes sont indiqués par les nombres 0, 1, 2 et 3.

Une *surface de translation* est un nombre fini de polygones P_1, P_2, $\ldots$, P_n dont on colle deux à deux les côtés en respectant la règle suivante : deux côtés collés doivent avoir la même longueur, la même direction et des vecteurs normaux opposés. Le *flot linéaire dans la direction* θ d'une surface de translation consiste à suivre la direction θ à vitesse 1 (voir l'exemple de la figure 1.5). Lorsqu'on ne précise pas la direction θ, nous considérons qu'il s'agit de la direction verticale $\theta = \pi/2$.

L'intérêt du flot d'une surface de translation par rapport à celui d'un billard est qu'il n'y a plus d'obstacle et donc plus de rebond : les trajectoires vont en ligne de droite. Cette propriété des surfaces de translation permet de *renormaliser le temps* via une action géométrique (voir le chapitre 2) : le flot à vitesse c (imaginer c très grand) dans une surface de translation correspond à un flot à vitesse 1 d'une autre surface de translation. Il est ainsi possible d'étudier la trajectoire d'une particule à de grandes échelles temporelles et d'en comprendre les phénomènes asymptotiques en étudiant une autre surface. Les résultats que nous présentons dans la section 1.5 proviennent de l'étude de cette renormalisation.

Construisons une surface de translation à partir du billard quotient $\overline{V}(a,b)$ de la section précédente, par une procédure de dépliage dite de *Fox-Kershner* [FK36] ou de *Katok-Zemliakov* [KZ75] (voir [MT02] pour la construction générale). Lorsque la particule rencontre un obstacle elle passe désormais dans une copie réfléchie du billard. Dans le cas du vent dans les arbres, quatre copies du billard initial sont nécessaires et correspondent aux quatre éléments du groupe $K = \{1, h, v, hv\}$ (voir la figure 1.6). Nous nommons cette surface $\mathrm{X}(a,b)$. Chaque copie du billard dans la surface $\mathrm{X}(a,b)$ correspond à une direction dans le billard initial ; autrement dit, chaque fois que la particule est dans la direction $g(\theta)$ dans le billard $\overline{V}(a,b)$, la trajectoire dépliée dans $\mathrm{X}(a,b)$ est dans une même copie (comparer les figures 1.3b et 1.6).

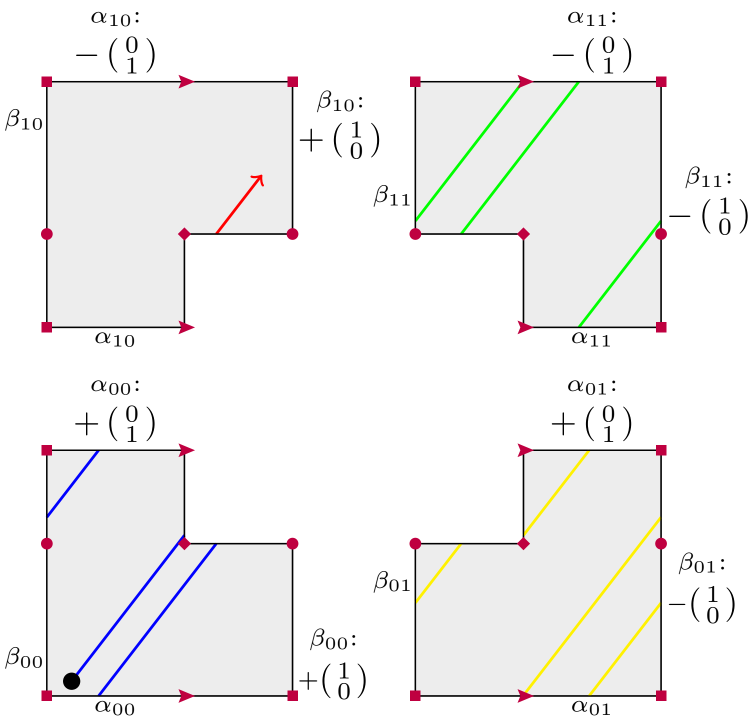


FIGURE 1.6 – Dépliage du billard quotient $\overline{V}(a,b)$ par la procédure de Katok-Zemliakov : la surface de translation $\mathrm{X}(a,b)$. Il faut identifier les côtés qui portent les mêmes étiquettes (h_{xy} ou v_{xy}) et ceux qui ont les mêmes sommets à leurs extrémités. Les vecteurs indiquent comment relever la trajectoire dans la table de billard $V(a,b)$ (cocycle du vent dans les arbres). Le morceau de trajectoire tracé sur cette figure est le dépliage de la trajectoire dans le billard de la figure 1.3a.

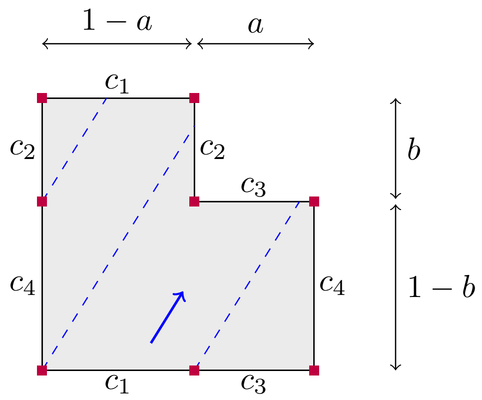
Remarquons que la procédure de dépliage a fait disparaître les obstacles et apparaître quatre *points singuliers* correspondant à certains sommets du billard quotient (sur la figure 1.6 il s'agit des points violets de différentes formes). Si une particule fait un petit cercle autour d'un de ces points, elle fait trois tours avant que sa trajectoire ne retourne à sa position initiale ! Autrement dit, un cercle de rayon r suffisamment petit autour de ce point a une circonférence de $6\pi r$ et non pas $2\pi r$. Ce sommet est une *singularité conique d'angle* 6π. Tout comme dans le billard quotient dans lequel le rôle des singularités étaient joués par les coins, une trajectoire qui rencontre une singularité conique n'a pas de prolongement.

À partir d'une trajectoire dans la surface $\mathrm{X}(a,b)$, il est possible de reconstruire la trajectoire du billard infini $V(a, b)$. Nous appelons encore *cocycle du vent dans les arbres* cette procédure et la notons toujours $f^{(T)}$. Comme dans le billard quotient (figure 1.3a) le cocycle compte les intersections avec certains côtés. Cependant, le lecteur sera attentif aux signes produits par les réflexions du billard en particulier sur la figure 1.6.

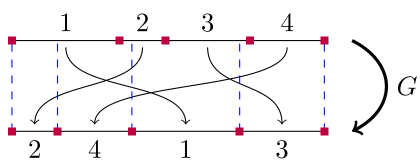
1.4 Flots linéaires de $\mathrm{X}(a, b)$ et échanges d'intervalles

Dans cette section, nous présentons une discrétisation du flot linéaire de la surface $\mathrm{X}(a, b)$: au lieu d'étudier un flot continu nous considérons l'itération d'une fonction G. L'application G que nous obtenons s'appelle un *échange d'intervalles*. Cette discrétisation permet de voir le cocycle du vent dans les arbres comme une *somme de Birkhoff*.

Nous commençons avec la surface $\mathrm{L}(a, b)$ afin de simplifier notre propos (voir la figure 1.7).



(a) Le flot dans la surface $\mathrm{L}(a, b)$ dans la direction θ. La direction du flot est indiquée par une flèche. Les lignes en pointillés sont les trajectoires (dans le futur).



(b) L'échange d'intervalles obtenu comme application de Poincaré du flot de la figure 1.7a. Les points où est découpé l'intervalle correspondent exactement aux singularités coniques.

FIGURE 1.7 – Le flot de la surface $\mathrm{L}(a, b)$ dans une direction θ et l'échange d'intervalles associé.

Discrétisons le flot de la surface $\mathrm{L}(a, b)$ dans la direction θ avec une *application de Poincaré*. Il faut choisir un segment transverse au flot ; par exemple la réunion I des côtés c_1, c_2, c_3 et c_4 du polygone de la figure 1.7. Étant donné un point p sur ce segment, on note $G(p) = \phi^{\theta}_{T(p)}$ le point du segment obtenu en suivant le flot linéaire à partir de p et en s'arrêtant lorsqu'on l'on croise à nouveau le segment. Le temps $T(p)$ est le temps nécessaire à la particule p pour traverser le polygone de la figure 1.7 et s'appelle *temps de premier retour*.

L'application $G : I \to I$ que nous avons construite est un *échange d'intervalles*. En effet, appliquer G revient à découper le segment I et à réorganiser les morceaux (voir la figure 1.7b). Les longueurs qu'il faut attribuer aux côtés pour conserver les longueurs dans la réorganisation sont les *longueurs transverses du flot* qui sont ici :

$$\lambda_1 = (1-a)\ \sin\theta \qquad \lambda_2 = b\ \cos\theta \qquad \lambda_3 = a\ \sin\theta \qquad \lambda_4 = (1-b)\ \cos\theta. \tag{1.1}$$

Pour encoder la réorganisation, il suffit de se donner une permutation, ici

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

Se donner π et λ suffit à décrire l'application de Poincaré G. Plus généralement étant donné un entier d, un vecteur de longueurs $\lambda \in \mathbb{R}_+^d$ et une permutation $\pi \in S_d$ on associe un échange de d intervalles $G_{\pi,\lambda}$.

Pour construire une application de Poincaré pour le flot linéaire de la surface $\mathrm{X}(a,b)$, il vaut mieux utiliser 4 segments transverses plutôt qu'un. Afin de garder plus de symétrie nous décidons de ne pas prendre exactement les côtés des polygones de la figure 1.6. En effet, quitte à découper et recoller quelques morceaux, la surface $\mathrm{X}(a,b)$ est constituée de 4 polygones en forme de « L » (voir figure 1.8). Les côtés de ces quatre « L » forment 4 segments dont la réunion est $I = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$.

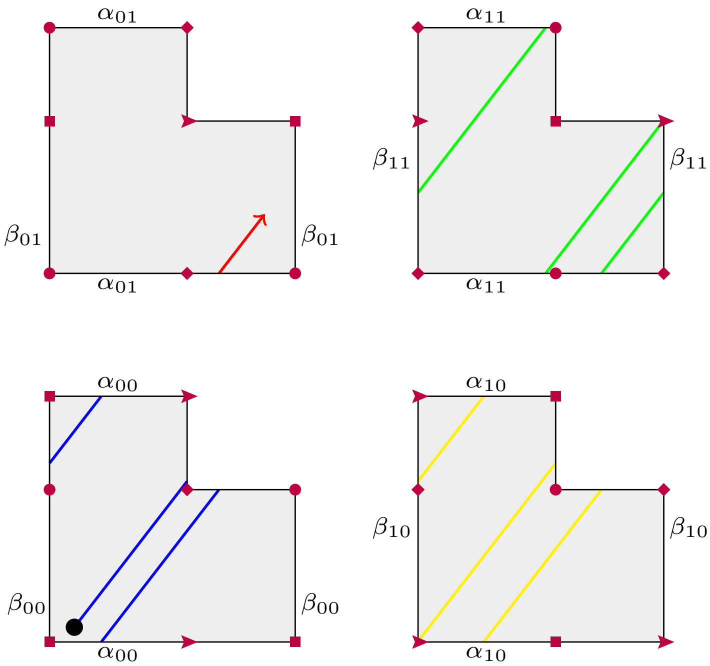


FIGURE 1.8 – La surface $\mathrm{X}(a,b)$ (figure 1.6) vue comme 4 copies de la surface $\mathrm{L}(a,b)$ (figure 1.7).

L'application de Poincaré du flot linéaire de $\mathrm{X}(a,b)$ sur I est un échange d'intervalles. Les quatre permutations qui décrivent le découpage des 4 intervalles sont

$$\pi_{10} = \begin{pmatrix} 1_{10} & 2_{10} & 3_{10} & 4_{10} \\ 2_{11} & 4_{10} & 1_{10} & 3_{00} \end{pmatrix} \qquad \pi_{11} = \begin{pmatrix} 1_{11} & 2_{11} & 3_{11} & 4_{11} \\ 2_{10} & 4_{11} & 1_{11} & 3_{01} \end{pmatrix}$$

$$\pi_{00} = \begin{pmatrix} 1_{00} & 2_{00} & 3_{00} & 4_{00} \\ 2_{01} & 4_{00} & 1_{00} & 3_{10} \end{pmatrix} \qquad \pi_{01} = \begin{pmatrix} 1_{01} & 2_{01} & 3_{01} & 4_{01} \\ 2_{00} & 4_{01} & 1_{01} & 3_{11} \end{pmatrix}.$$

Les longueurs sont les mêmes que pour la surface $\mathrm{L}(a,b)$ (voir (1.1)).

Les sauts effectués par le cocycle du vent dans les arbres, s'identifient à une fonction $f : I \to \mathbb{Z}^2$ constante sur chacun des 16 sous-intervalles de cet échange d'intervalles et dont les valeurs sont parmi

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Le cocycle du vent dans les arbres correspond alors à la *somme de Birkhoff* de cette fonction. Au temps correspondant au N-ème retour, le domaine fondamental dans lequel se trouve la particule est

$$S_N(f,p) = f(p) + f(G(p)) + \ldots + f(G^{N-1}(p)).$$

Remarquons par ailleurs, que le temps de N-ème retour est lui aussi une somme de Birkhoff.

1.5 Diffusion et récurrence du vent dans les arbres

La description du vent dans les arbres est donnée par une somme de Birkhoff le long des orbites d'un échange d'intervalles. Cette description est celle qui sera utilisée dans les chapitres suivants. Dans cette section, nous anticipons sur le développement des techniques relatives aux surfaces de translation et donnons quelques résultats sur la dynamique du vent dans les arbres. Notre travail porte sur les propriétés de *diffusion* et de *récurrence* qui font l'objet des articles présentés respectivement dans les annexe A et B.

Soit a et b deux paramètres réels entre 0 et 1. On note ϕ_T^θ le flot du billard $V(a,b)$ dans la direction θ. La *diffusion* est le terme qui désigne la vitesse à laquelle une particule s'éloigne de sa position initiale. Nous avons vu que la distance d'une particule à son point de départ $d(p, \phi_T^\theta(p))$ s'approche par une somme de Birkhoff $S_N(G,f,p)$ d'une fonction f le long de l'orbite d'un point p pour l'échange d'intervalle G. Ainsi, nous essayons d'estimer la taille d'une somme de Birkhoff. Le premier terme d'approximation de cette quantité est donné par la limite du rapport $d(p, \phi_T^\theta(p))/T$ qui correspond aux *moyennes de Birkhoff* $S_N(G,f,p)/N$. Cette convergence est assurée sous la condition d'*unique ergodicité* qui a été démontrée par S. Kerckhoff, H. Masur et J. Smillie [KMS86]. Plus précisément :

F*ixons a et b. Pour un angle θ générique, pour tout point p, on a :*

$$\lim_{T\to\infty} \frac{d(p, \phi_T^\theta(p))}{T} = 0.$$

Ainsi la vitesse à laquelle une particule explore le billard $V(a,b)$ est sous-linéaire. Dans l'annexe B, nous calculons un exposant qui affine ce résultat.

P*our des paramètres a, b et θ génériques, pour tout point p, on a :*

$$\limsup_{T\to\infty} \frac{\log d(p, \phi_T^\theta(p))}{\log T} = 2/3.$$

Ce coefficient $2/3$ est lié à l'action de $\mathrm{SL}(2,\mathbb{R})$ sur les surfaces de translation et correspond à la vitesse de renormalisation du cocycle du vent dans les arbres. Plus précisément, il s'agit d'un *exposant de Lyapunov du cocycle de Kontsevich-Zorich* dont nous faisons le calcul explicite dans la section 4.3. Nous finissons par deux remarques :

- notre théorème de diffusion ne s'applique qu'à des paramètres a et b génériques, contrairement au théorème de Birkhoff ;
- dans notre résultat, il s'agit d'une limite supérieure et non pas d'une limite.

La diffusion concernait la taille maximale de la quantité $d(x, \phi_T^\theta(x))$. De manière complémentaire, on cherche à savoir si une particule va revenir arbitrairement proche de son point de départ ou si au contraire elle va s'en aller à l'infini. Dans le premier cas, on dit que la trajectoire est *récurrente* et dans le second *divergente*.

Les seuls résultats que nous obtenons pour la récurrence, concerne des cas où la surface $\mathrm{X}(a,b)$ est particulièrement symétrique ; plus précisément lorsque c'est une *surface de Veech* (voir la partie 4 pour une définition précise). Les travaux de K. Calta [Cal04] et C. McMullen [McM03] listent les paramètres a et b pour lesquels cette condition est réalisée :

> L*a surface* $\mathrm{X}(a,b)$ *est une surface de Veech si et seulement si* a *et* b *vérifient l'une des deux conditions suivantes :*
>
> *1.* a *et* b *sont rationnels ;*
>
> *2. il existe* x *et* y *deux nombres rationnels et* D *un entier positif sans facteur carré tel que :*
>
> $$\frac{1}{1-a} = x + y\sqrt{D} \qquad \textit{et} \qquad \frac{1}{1-b} = (1-x) + y\sqrt{D}.$$

P. Hubert, S. Lelièvre et S. Troubetzkoy [HLT] démontrent le théorème suivant :

> P*our des paramètres rationnels* $a = p/q$ *et* $b = r/s$ *avec* p, r *impairs et* q, s *pairs, pour un angle* θ *générique, le flot* ϕ_t^θ *est récurrent.*

Nous expliquons brièvement la preuve de ce résultat. Remarquons que si a et b sont rationnels alors pour tout angle de la forme $\theta = \arctan(p/q)$ avec $p/q \in \mathbb{Q}$, un phénomène périodique apparaît (les trajectoires sont périodiques dans le billard quotient $\overline{V}(a,b)$). C'est le cas par exemple de la direction horizontale et verticale dans lesquelles une famille de trajectoires fait des rebonds entre deux obstacles et une autre famille diverge à vitesse linéaire (*direction mixte*). Dans la direction $\pi/4$ avec $a = b = 1/2$ toutes les trajectoires sont périodiques dans $V(a,b)$ (*direction périodique*). Les différentes alternatives sur la forme des trajectoires dans les directions rationnelles sont analysées en utilisant la géométrie des surfaces de translation de genre 2. La preuve est composée de trois ingrédients :

- Une direction irrationnelle « bien approchée » par des directions rationnelles périodiques est récurrente.
- Pour les paramètres a et b de l'énoncé il existe « beaucoup » de directions rationnelles périodiques dans $V(a,b)$ et une direction θ générique est bien approchée par les dirèctions périodiques.

En contrepoint à ce théorème et dans un cadre plus général, nous démontrons dans l'annexe A, l'existence de trajectoires divergentes dans les billards $V(a,b)$:

> P*our tout* a *et* b *tels que* $\mathrm{X}(a,b)$ *est une surface de Veech, il existe un ensemble* $\Lambda \subset S^1$ *de dimension de Hausdorff positive tel que, pour tout angle* $\theta \in \Lambda$*, toute orbite du flot* ϕ_T^θ *dans est divergente.*

La démonstration de notre résultat repose sur l'induction de Ferenczi-Zamboni qui est une version combinatoire du flot de Teichmüller. Elle consiste, comme dans le théorème précédent sur la récurrence, à approcher une pente θ par des pentes rationnelles mais qui sont cette fois de type mixte.

Chapitre 2

Renormalisation du flot linéaire d'une surface de translation

Dans le chapitre précédent, nous avons vu que les trajectoires d'un billard rationnel pouvaient être vues comme des orbites du flot linéaire d'une surface de translation. Cette description nous a permis de reformuler certains problèmes dynamiques du vent dans les arbres en terme du flot linéaire de la surface $\mathrm{X}(a, b)$. Dans ce chapitre, nous introduisons l'action (géométrique) du *flot de Teichmüller* sur une surface de translation qui correspond à une renormalisation (temporelle) de son flot linéaire. Les propriétés de l'orbite d'une surface sous l'action du flot de Teichmüller reflète ses propriétés dynamiques.

G. Rauzy [Rau79] introduit une procédure combinatoire pour renormaliser les échanges d'intervalles qui étendue par W. Veech [Vee82] pour comprendre la dynamique du flot de Teichmüller sur les espaces de surfaces de translation. Cette induction, dite de *Rauzy-Veech*, effectue une suite de découpage-recollage des polygones définissant une surface de manière à compenser les déformations du flot de Teichmüller.

Les sommes de Birkhoff que l'on considère au-dessus des flots linéaires, comptent les intersections d'une orbite avec les côtés. À une nouvelle représentation polygonale est associée un changement de base dans ce comptage appelé *cocycle de Kontsevich-Zorich*. Ce dernier est l'objet d'étude du chapitre 3.

2.1 Flot de Teichmüller et induction de Rauzy

Dans la section 1.4, nous avons vu que les applications de Poincaré du flot linéaire d'une surface de translation sont des échanges d'intervalles. Ces derniers, à une renormalisation près du temps, encodent la dynamique du flot linéaire. Dans cette partie, ils jouent un rôle essentiel : les surfaces de translation vont être vues comme des *suspensions* d'échanges d'intervalles.

Rappelons qu'un échange d'intervalles est déterminée par la donnée d'une permutation π (la façon dont on mélange les intervalles) et un vecteur λ (les longueurs des intervalles découpés). Suivant [MMY05] et [Buf06], afin de pouvoir suivre les intervalles découpés, nous donnons des noms à chacun d'eux. Une *permutation étiquetée* est un couple $\pi_t, \pi_b : \mathcal{A} \to \{1, 2, \ldots, d\}$ de bijections d'un ensemble fini $\mathcal{A}$ appelé *alphabet* que l'on note :

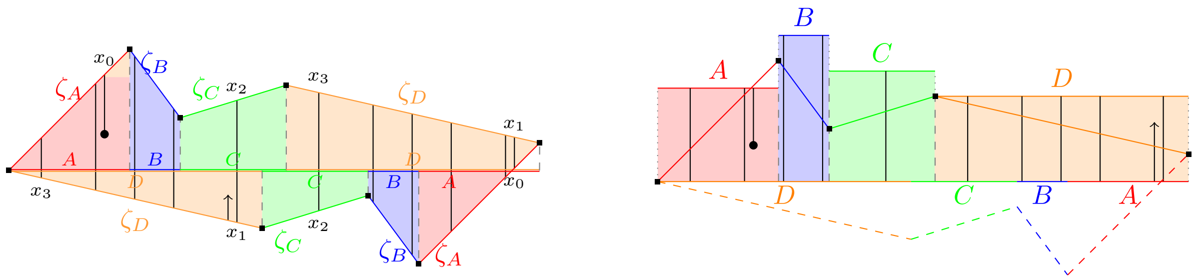
$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \ldots & \pi_t^{-1}(d) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \ldots & \pi_b^{-1}(d) \end{pmatrix}.$$

La ligne du haut, *top*, (resp. du bas, *bottom*) correspond à l'ordre des intervalles du haut (resp. du bas).

Soit $T = T_{\pi,\lambda}$ un échange d'intervalles étiqueté avec $\pi = (\pi_t, \pi_b)$ une permutation irréductible[1]. Nous voulons faire la construction inverse de l'application de premier retour (voir la section 1.4) : construire une surface à partir de l'échange d'intervalles. Ce dernier donne les coordonnées horizontales de la surface, et il suffit donc de se donner des hauteurs. Une *donnée de suspension* pour $T_{\pi,\lambda}$ est un vecteur $\zeta = \lambda + i\tau \in \mathbb{C}^d$ tel que pour tout $k < d$:

$$\sum_{i=1}^{k} \tau_{\pi_t^{-1}(i)} > 0 \qquad \text{et} \qquad \sum_{i=1}^{k} \tau_{\pi_b^{-1}(i)} < 0.$$

À une donnée de suspension, on associe une ligne brisée L_t (resp. L_b) en concaténant les vecteurs $\pi_t^{-1}(k)$ (resp. $\pi_b^{-1}(k)$) pour $k = 1, \ldots, d$. La surface $S(\pi, \zeta)$ est construite à partir du polygone défini par les lignes brisées L_t et L_b dans lequel on identifie les côtés définis par le même vecteur (voir la figure 2.1a). On peut également construire la même suspension en utilisant les *rectangles cousus* de Veech [Vee82] (voir la figure 2.1b).



(a) La suspension construite avec les lignes brisées L_t et L_b. L'unique singularité conique de cette suspension est indiquée par des carrés noirs.

(b) Les rectangles cousus. La couture s'arrête à la première singularité conique rencontrée.

FIGURE 2.1 – Les deux constructions d'une suspension de la permutation $\binom{A\ B\ C\ D}{D\ C\ B\ A}$. Nous avons tracé une orbite de son flot linéaire en noir dont le codage est $w = A\,D\,C\,D\,A\,D\,B\,D\,B\,D\,A\,D$.

Le *flot de Teichmüller* est l'action du sous-groupe à un paramètre $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ sur les données de suspensions. Ce dernier contracte la direction verticale et dilate la direction horizontale. En particulier, une orbite du flot linéaire de longueur 1 dans la surface $g_t \cdot S$ est une orbite du flot linéaire de longueur e^t dans la surface S (voir la figure 2.2). Le flot de Teichmüller, pris isolément,

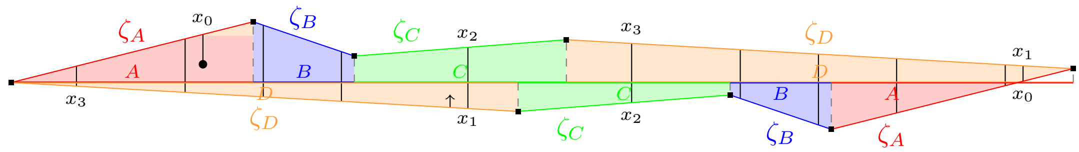


FIGURE 2.2 – Action du flot de Teichmüller sur la suspension de la figure 2.1. L'orbite du flot linéaire, en noir, est maintenant beaucoup plus courte.

n'a que peu d'intérêt : la surface s'allonge indéfiniment. Afin de pouvoir mesurer son effet, on opère des découpages de la surface de manière à conserver une longueur raisonnable ;

[1] Une permutation (π_t, π_b) est dite *réductible* si il existe $k < d$ tel que $\pi_t(\{1, \ldots, k\}) = \pi_b(\{1, \ldots, k\})$. Une permutation qui n'est pas réductible est dite *irréductible*.

Soit $T = T_{\pi,\lambda}$ un échange d'intervalles sur l'intervalle I. Considérons les deux sous-intervalles respectivement à droite du domaine et de l'image de T dont les étiquettes sont $\alpha_t = \pi_t^{-1}(d)$ et $\alpha_b = \pi_b^{-1}(d)$. Définissons le sous-intervalle de I dans lequel on enlève le plus petit de ces deux intervalles ; autrement dit $J = [0, |\lambda| - \min(\lambda_{\alpha_t}, \lambda_{\alpha_b})]$. L'*induction de Rauzy* est l'application qui, à l'échange d'intervalles T, associe l'application de premier retour de T sur J (voir la figure 2.3). G. Rauzy [Rau79] démontre que cette application de premier retour est un échange d'intervalles avec le même nombre de sous-intervalles que T.

La modification de la permutation et du vecteur de longueurs ne dépendent que du type d'induction. Notons, (π, λ) les données d'un échange d'intervalles et (π', λ') les données obtenues après une induction de Rauzy. Si l'induction est de type top alors :

$$\text{pour } \alpha \neq \alpha_t \ \lambda'_\alpha = \lambda_\alpha \qquad \text{et} \lambda'_{\alpha_t} = \lambda_{\alpha_t} - \lambda_{\alpha_b}.$$

Le cas de l'induction bottom est identique. Dans les deux cas, la matrice $M(\pi, \lambda)$ telle que $M(\pi, \lambda)\lambda' = \lambda$ est une matrice élémentaire.

La permutation π' obtenue à partir de π après une induction de Rauzy ne dépend que du type (top ou bottom) de l'induction. On obtient ainsi deux opérations combinatoires R_t et R_b agissant sur l'ensemble des permutations irréductibles. Par exemple, à partir de la permutation étiquetée $\pi = \binom{A\,B\,C\,D}{D\,C\,B\,A}$ on obtient les deux permutations :

$$R_t(\pi) = \begin{pmatrix} A\,B\,C\,D \\ D\,A\,C\,B \end{pmatrix} \qquad \text{et} \qquad R_b(\pi) = \begin{pmatrix} A\,D\,B\,C \\ D\,C\,B\,A \end{pmatrix}.$$

Ces deux configurations apparaissent dans la figure 2.3 à la 1re et 5e étape respectivement.

L'*induction de Rauzy-Veech* est l'application qui à $S = S(\pi, \zeta)$ associe la suspension $S' = S(\pi', \zeta')$ où l'on remplace simplement λ par ζ dans l'induction de Rauzy. La surface S est canoniquement isomorphe à S' car elle s'obtient par découpage et collage (voir sur la figure 2.3).

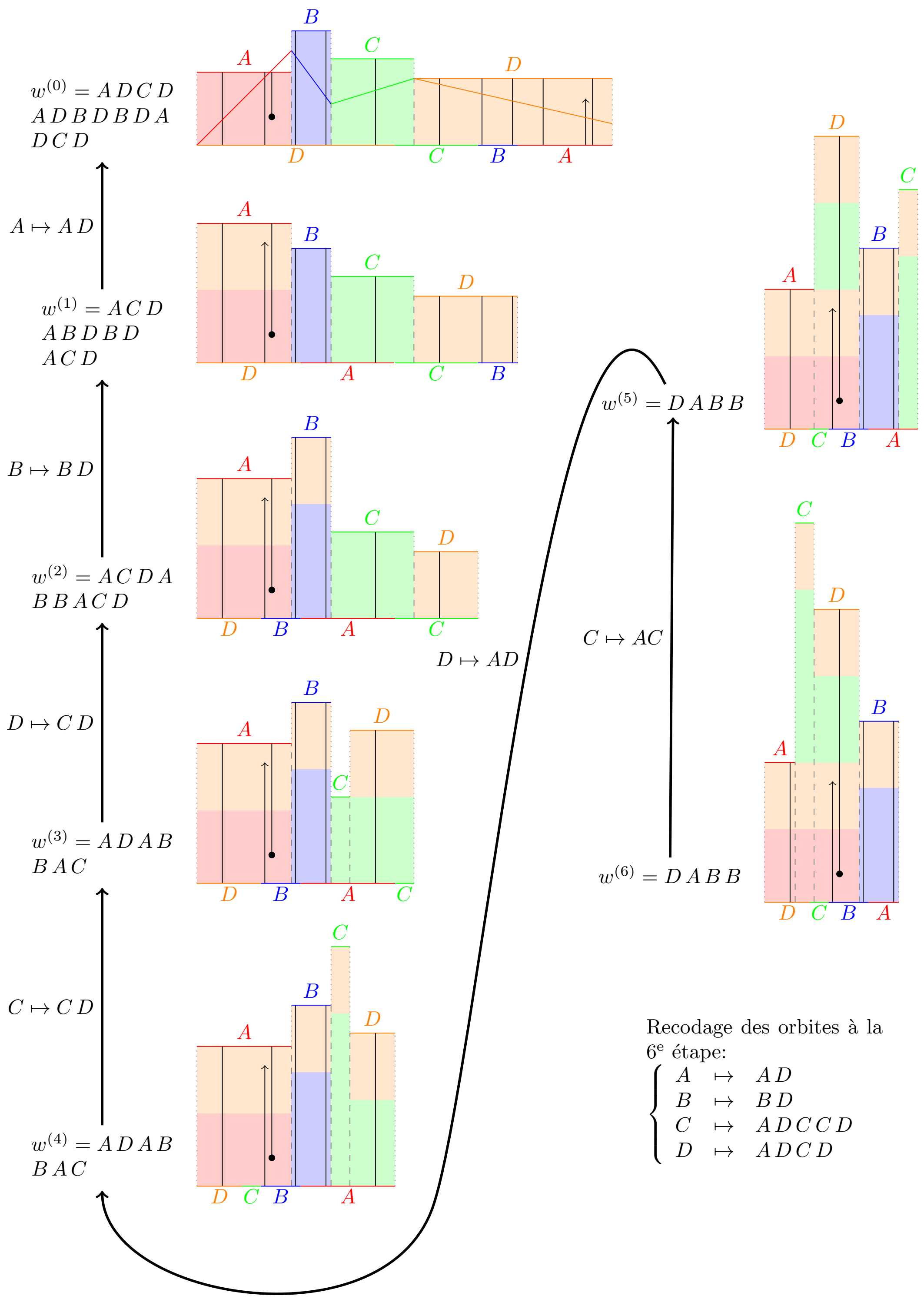


FIGURE 2.3 – Six inductions de Rauzy-Veech successives vues sur les rectangles cousus de Veech. La surface de départ est la même que sur la figure 2.1. Les types d'induction sont successivement t, t, b, t, b.

Considérons une suspension $S = S(\pi,\zeta)$ et son image au temps t par le flot de Teichmüller $g_t \cdot S$. Notons $(\pi^{(n)}, \zeta^{(n)})$ la suite de données obtenues par l'induction de Rauzy-Veech à partir de (π,ζ). Comme l'induction de Rauzy commute avec le flot de Teichmüller, on peut effectuer des inductions de Rauzy-Veech pour la suspension $g_t \cdot S$ jusqu'à ce qu'elle soit de longueur raisonnable : on choisit l'étape d'induction n telle que $|\lambda^{(n)}| \geq e^{-t} > |\lambda^{(n+1)}|$ où $\lambda^{(n)}$ est la partie réelle de $\zeta^{(n)}$ et $|\lambda| = \lambda_A + \lambda_B + \lambda_C + \lambda_D$ désigne la somme des coordonnées. Autrement dit, la longueur de l'intervalle vue sur $g_t \cdot S$, soit $e^t \cdot |\lambda^{(n)}|$, est à peu près de taille 1 (voir la figure 2.4).

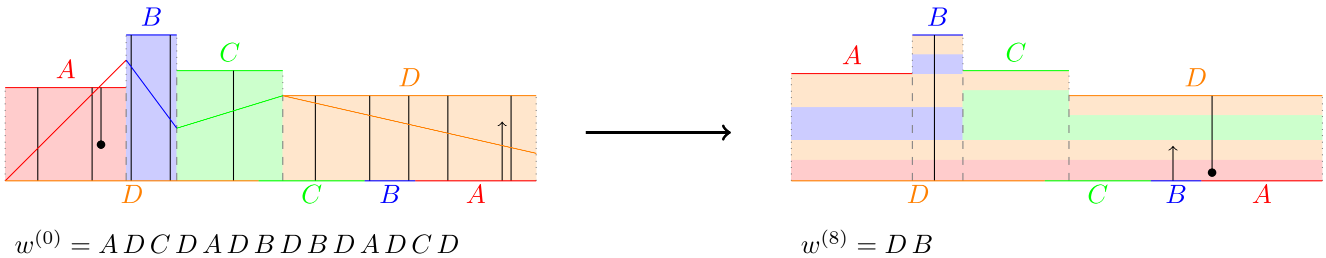


FIGURE 2.4 – Action simultanée du flot de Teichmüller au temps $t_0 \simeq 4.329$ et de l'induction de Rauzy sur la suspension de la figure 2.1 (voir aussi les figures 2.2 et 2.3). La trajectoire longue du flot linéaire de la surface initiale (sur la gauche) apparaît comme une trajectoire courte sur la droite. Les deux polygones qui définissent les surfaces sont les mêmes.

Afin de comprendre comment passer d'une somme de Birkhoff sur S à une somme de Birkhoff sur $g_t \cdot S$, nous introduisons le *cocycle de Kontsevich-Zorich*. Soit (π,λ) les données d'un échange d'intervalles et $(\pi^{(n)}, \lambda^{(n)})$ la suite de données obtenues par l'induction de Rauzy. À chaque étape de l'induction de Rauzy, le vecteur $\lambda^{(k)}$ est modifié par une matrice élémentaire $M_k = M(\lambda^{(k)}, \pi^{(k)}$ qui vérifie $\lambda = M_0 M_1 \ldots M_{n-1} \lambda^{(n)}$. A un temps $t \geq 0$, on associe le produit de matrice $M^{(t)}(\pi,\lambda) = M_0 M_2 \ldots M_{n-1}$ obtenue pour l'induction de Rauzy où, comme ci-dessus, n est l'entier tel que $|\lambda^{(n)}| \geq e^{-t} > |\lambda^{(n+1)}|$. Sur la figure 2.4, la matrice M associée à la huitième étape de l'induction de Rauzy (ou le cocycle de Kontsevich-Zorich au temps $t_0 \simeq 4.329$, valeur propre dominante de M) est :

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}.$$

L'échange d'intervalles qui nous sert d'exemple est un peu particulier car $(\pi^{(8)}, \lambda^{(8)})$ est égal aux données initiales (π,λ) modulo le facteur de dilatation t_0. Autrement dit, la suite des inductions de Rauzy est périodique : on dit que l'échange d'intervalles est *autosimilaire*. Le vecteur de longueurs λ est le vecteur propre de Perron-Frobenius de la matrice M :

$$\lambda_A \simeq 0.2278 \qquad \lambda_B \simeq 0.0953 \qquad \lambda_C \simeq 0.1997 \qquad \lambda_D \simeq 0.4772.$$

Pour obtenir une surface autosimilaire, il faut choisir comme donnée de hauteur τ un vecteur propre associé à la plus petite valeur propre de M $(1/t_0 \simeq 0.2278)$. Par exemple :

$$\tau_A = 1 \qquad \tau_B \simeq -0.5642 \qquad \tau_C \simeq 0.2693 \qquad \tau_D \simeq -0.4772.$$

Notons $\zeta = \lambda + i\tau$ et $S = S(\pi,\zeta)$ la suspension. Par construction, la surface $g_{t_0} \cdot S$ est isomorphe à S.

La substitution qui décrit le recodage des orbites de $g_{t_0} \cdot S$ à S est

$$\begin{cases} A & \mapsto & A\,D\,B\,D \\ B & \mapsto & A\,D\,B\,D\,B\,D \\ C & \mapsto & A\,D\,C\,C\,D \\ D & \mapsto & A\,D\,C\,D \end{cases}.$$

Elle peut se lire directement sur la suspension : sur la partie droite de la figure 2.4, le rectangle nommé A est composé successivement des couches A, D, B et D de la figure de gauche. Il en est de même pour les autres lettres. La matrice M est une version simplifiée de cette substitution qui ne s'occupe pas de l'ordre de lettres : chaque colonne correspond aux nombres de chacune des lettres dans l'image correspondante. Comme l'induction de Rauzy découpe l'intervalle en ne conservant que son extrémité gauche, le point fixe de cette substitution

$$w = \sigma^{\infty}(A) = A\,D\,B\,D\,A\,D\,C\,D\,A\,D\,B\,D\,B\,D\,A\,D\,C\,D\,A\,D\,A\,D\,B\,D\,A\,D\,C\,\ldots$$

est le codage de l'orbite de ce point extrémal. Le lecteur pourra vérifier que le codage de la trajectoire finie présentée dans la figure 2.3 est un *facteur* du mot w ; autrement dit, qu'il apparaît dans w.

2.2 Classes de Rauzy et composantes de strates

Dans la section précédente nous avons décrit comment suivre la trajectoire d'une surface de translation sous l'action du flot de Teichmüller en utilisant l'induction de Rauzy-Veech. Cependant, nous n'avons pas décrit dans quel espace cette déformation avait lieu ; il s'agit des *composantes de strates* dont une version combinatoire est donnée par les *diagrammes de Rauzy*.

À l'induction de Rauzy, on associe naturellement un graphe orienté dont les sommets sont les permutations irréductibles et les arêtes sont l'action des opérations R_t et R_b correspondant aux opérations top et bottom de l'induction de Rauzy. Chaque composante connexe de ce graphe est appelé un *diagramme de Rauzy* (voir l'exemple de la figure 2.5). L'ensemble des sommets d'un diagramme de Rauzy s'appelle une *classe de Rauzy*. Si π est une permutation réduite (resp. étiquetée) sa classe de Rauzy est appelée *classe de Rauzy réduite* (resp. *classe de Rauzy étiquetée*).

Les diagrammes de Rauzy sont en correspondance avec des objets plus géométriques : des *composantes connexes de strates*. A une surface de translation S, on lui associe la liste $(\kappa_1, \ldots, \kappa_s)$ des degrés des singularités : le degré κ_i correspond à un angle $2(\kappa_i + 1)\pi$. Le *genre* g de la surface est l'entier qui vérifie la formule :

$$\kappa_1 + \kappa_2 + \ldots + \kappa_s = 2g - 2.$$

Fixons un entier $g \geq 1$. Il existe un *espace des modules*, $\mathcal{H}_g$ dont chaque point correspond à une classe d'isomorphisme de surface de translation de genre g et d'aire 1. Par isomorphisme, on entend égal à découpage et recollage près. L'espace $\mathcal{H}_g$ est découpé en *strates*. Pour chaque données $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_s)$ de degrés de singularités coniques, on considère la strate $\mathcal{H}_g(\kappa)$ constituée des surfaces de genre g dont la liste des degrés des singularités conique est κ. Le g dans la notation $\mathcal{H}_g(\kappa)$ est donc superflu, on note alors $\mathcal{H}(\kappa)$. Pour chaque genre g, l'espace des modules $\mathcal{H}_g$ est la réunion disjointe de strates $\mathcal{H}(\kappa_1, \ldots, \kappa_s)$ où $\kappa_1 + \ldots + \kappa_s = 2g - 2$. Par exemple, en genre 2, il y a deux strates : $\mathcal{H}(2)$ et $\mathcal{H}(1,1)$. En genre 3, il y en a cinq : $\mathcal{H}(4)$, $\mathcal{H}(3,1)$, $\mathcal{H}(2,2)$, $\mathcal{H}(2,1,1)$ et $\mathcal{H}(1,1,1,1)$.

Étant donnée une permutation irréductible π et une suspension $S = S(\pi, \zeta)$, la strate contenant S se calcule de la manière suivante. On tourne dans le sens trigonométrique autour des singularités de la surface qui correspondent à chaque extrémités des vecteurs ζ_α des lignes brisées L_t et L_b. Il faut alors compter le nombre de tour que l'on effectue. Lorsqu'on est sur la ligne L_t et que l'on tourne,

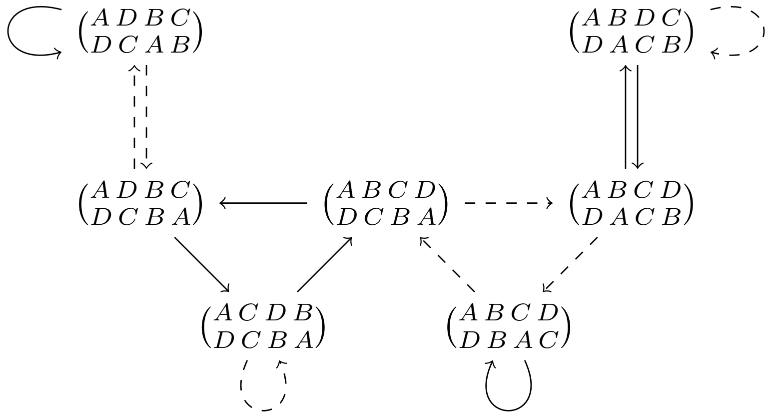
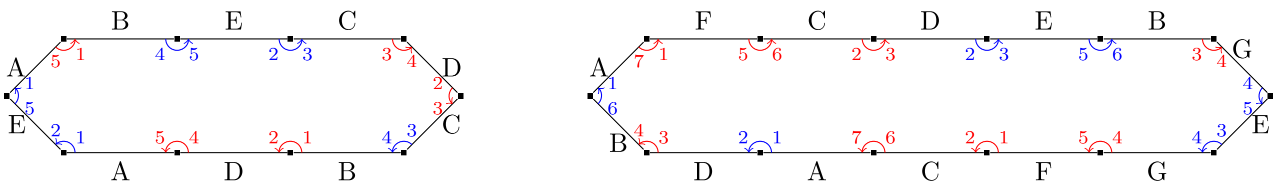


FIGURE 2.5 – Diagramme de Rauzy de la permutation $\pi = \binom{A\,B\,C\,D}{D\,C\,B\,A}$. L'opération R_t correspond aux arêtes hachurées et R_b aux arêtes pleines.

on passe du vecteur $\pi_t^{-1}(i)$ au vecteur $\pi_t^{-1}(i+1)$. Lorsqu'on est sur la ligne L_b et que l'on tourne, on passe du vecteur $\pi_b^{-1}(i)$ au vecteur $\pi_b^{-1}(i-1)$. Il faut cependant prendre garde aux extrémités. Nous faisons ce calcul sur deux exemples dans la figure 2.6.



(a) Les deux singularités coniques sont d'angle 3π : la strate est $\mathcal{H}(1,1)$.

(b) Il y a une singularité conique d'angle 5π (en rouge) et une d'angle 3π (en bleu) : la strate est $\mathcal{H}(3,1)$.

FIGURE 2.6 – Calcul des strates associées aux deux permutations $\binom{A\,B\,E\,C\,D}{E\,A\,D\,B\,C}$ et $\binom{A\,F\,C\,D\,E\,B\,G}{B\,D\,A\,C\,F\,G\,E}$. La strate ne dépend que des permutations et non pas du choix particulier de la suspension, dans les deux cas, les dessins sont juste schématiques car il suffit de compter combien de fois un petit cercle autour d'une singularité croise une direction verticale.

Le flot de Teichmüller préserve l'aire des surfaces et le degré de singularités coniques et agit donc sur chaque strate. Comme il est continu, il préserve également les composantes connexes des strates. Le théorème fondateur de Masur-Veech [Mas82], [Vee82] assure que :

> *Chaque strate est de volume fini pour la mesure de Lebesgue*[2]*. Le flot de Teichmüller préserve cette mesure et est ergodique.*

Comme le flot de Teichmüller agit ergodiquement sur les composantes connexes de strates, la construction des classes de Rauzy suggère que l'on a une identification entre classes de Rauzy et composantes connexes de strates. Cependant, il faut prendre garde au fait que l'induction de

[2] Il existe une mesure naturelle sur les strates. Cette mesure peut-être vue sur les suspensions d'échanges d'intervalles $\zeta = \lambda + i\tau$ en désintégrant la mesure de Lebesgue $d\lambda \otimes d\tau$ des données de suspension sur l'ensemble des suspensions d'aires 1. L'aire d'une suspension est $\langle \lambda, h \rangle$ où h désigne le vecteur des hauteurs des rectangles dans la construction des rectangles cousus de Veech.

Rauzy-Veech fixe le point de la suspension situé à gauche de l'intervalle. Pour s'en affranchir, nous introduisons une opération supplémentaire $\iota : S_d \to S_d$ qui correspond à l'application échangeant le haut avec le bas et la gauche avec la droite des permutations :

$$\iota \begin{pmatrix} a_1 & a_2 & \dots & a_d \\ b_1 & b_2 & \dots & b_d \end{pmatrix} = \begin{pmatrix} b_d & b_{d-1} & \dots & b_1 \\ a_d & a_{d-1} & \dots & a_1 \end{pmatrix}.$$

Les *diagrammes de Rauzy étendus*, sont les composantes connexes du graphe dont les sommets sont les permutations irréductibles et les arêtes sont données par l'action de R_t, R_b etι. L'ensemble des sommets d'un diagramme de Rauzy étendu est une *classe de Rauzy étendue.*

Étant donnée une permutation π on note κ_π la partition de $2g-2$ donnée par les degrés des zéros, $\mathcal{C}(\pi) \subset \mathcal{H}(\kappa_\pi)$ la composante connexe de strate de toute suspension de π et enfin $m_l(\pi)$ le degré du zéro de la différentielle abélienne correspondant au côté gauche de l'intervalle. On note $\kappa'_\pi = \kappa_\pi \backslash \{m_l(\pi)\}$. W. Veech [Vee82, Vee89] démontre le théorème suivant[3] :

L*'application $\pi \to \mathcal{C}(\pi) \subset \mathcal{H}(\kappa_\pi)$ induit une bijection entre les classes de Rauzy étendues et les composantes connexes de strates.*
L'application $\pi \to (m_l(\pi), \mathcal{C}(\pi)) \subset \mathcal{H}(m_l(\pi); \kappa'_\pi)$ induit une bijection entre les classes de Rauzy et les composantes connexes de strates avec un degré marqué.

Les classes de Rauzy étendues donnent ainsi une version combinatoire des composantes connexes de strates. En utilisant cette approche, W. Veech et P. Arnoux, dans les années 80, ont construit les premiers exemples de strates non connexes (calcul des composantes connexes pour les strates $\mathcal{H}(4)$ et $\mathcal{H}(6)$, voir la remarque p. 159 de [Vee90]). Dans les années 90, M. Kontsevich et A. Zorich [Kon97], suite à des expérimentations sur les classes de Rauzy étendues, ont énoncé une conjecture sur cette classification en proposant des invariants géométriques. Ces mêmes auteurs [KZ03] démontrent quelques années plus tard cette conjecture :

[3] Ce résultat est étendu par C. Boissy [Boia] au cas des permutations généralisées et des différentielles quadratiques. Ces dernières sont des surfaces de translation dans lesquelles on autorise des inversions pour recoller les côtés.

L*es composantes connexes de la strate avec points marquées*[4] $\mathcal{H}(\kappa \cup (0^k))$ *sont en bijection avec les composantes connexes de la strate* $\mathcal{H}(\kappa)$ *via l'application naturelle qui consiste à oublier les points marqués.*
Les composantes connexes des strates de différentielles abéliennes sans point marqué pour le genre $g \geq 4$ *sont classifiées par les énoncés suivants :*

- *Les strates* $\mathcal{H}(g-1, g-1)$ *avec* g *impair et* $\mathcal{H}(2g-2)$ *pour tout* g *possèdent trois composantes connexes : une composante hyperelliptique et deux autres composantes identifiées par leur parité de structure spin (on appelle ces deux composantes* paire *et* impaire*).*

- *Les autres strates dont les degrés des zéros sont pairs* $\mathcal{H}(2m_1, 2m_2, \ldots, 2m_n)$ *possèdent deux composantes connexes qui sont distinguées par leur parité de structure spin (la* composante paire *et la* composante impaire*).*

- *La composante* $\mathcal{H}(g-1, g-1)$ *pour* g *pair a deux composantes connexes : une hyperelliptique et une autre baptisée* non-hyperelliptique*.*

- *Toutes les autres strates sont connexes.*

Pour les genres $g < 4$ *on a la classification suivante :*

- *Les strates* $\mathcal{H}_1(0)$*,* $\mathcal{H}_2(2)$ *et* $\mathcal{H}_2(1,1)$ *sont non vides et connexes.*

- *Les strates* $\mathcal{H}_3(4)$ *et* $\mathcal{H}_3(2,2)$ *possèdent deux composantes connexes une hyperelliptique et une autre dont la parité de structure spin est impaire. Les autres composantes de genre* 3 *sont connexes.*

Pour une partition κ de $2g-2$ ne contenant que des nombres pairs, on note $\mathcal{H}^{odd}(\kappa)$ et $\mathcal{H}^{even}(\kappa)$ les composantes impaire et paire de $\mathcal{H}(\kappa)$. Les composantes hyperelliptiques sont notées $\mathcal{H}^{hyp}(2g-2)$ et $\mathcal{H}^{hyp}(g-1, g-1)$. La composante non-hyperelliptique de $\mathcal{H}(g-1, g-1)$ pour g pair est notée $\mathcal{H}^{nonhyp}(g-1, g-1)$.

On déduit des deux théorèmes ci-dessus une classification des classes de Rauzy.

La surface $\mathrm{X}(a, b)$ du vent dans les arbres appartient à la strate $\mathcal{H}(2,2,2,2)$ qui possède deux composantes $\mathcal{H}^{odd}(2,2,2,2)$ et $\mathcal{H}^{even}(2,2,2,2)$. Le calcul de la parité de la structure spin[5] montre que $\mathrm{X}(a, b)$ est dans $\mathcal{H}^{odd}(2,2,2,2)$. Quant à la surface $\mathrm{L}(a, b)$, elle est dans la strate connexe $\mathcal{H}(2)$.

2.3 Comptage des permutations des classes de Rauzy

Dans l'annexe C, motivés par l'étude combinatoire de G. Rauzy [Rau79], nous établissons une formule pour la cardinalité des classes de Rauzy[6]. La correspondance entre les classes de Rauzy et les composantes de strates d'une part et la classification des composantes de strates d'autre part jouent un rôle essentiel dans notre preuve. Notons que pour l'induction de Ferenczi-Zamboni, le comptage

[4] Les points marqués correspondent à de fausses singularités coniques. Chaque strate avec points marqués est une strate de l'espace des modules $\mathcal{H}_{g,n}$ des classes d'isomorphismes de surface de translation avec n points marqués. Pour les strates, chaque point marqué est indiqué par un degré 0 dans le vecteur κ.

[5] Voir les articles [Joh80] et [KZ03].

[6] Signalons que différents travaux de comptage ont un lien avec la théorie de Teichmüller : J. Harer et D. Zagier [HZ86], en utilisant le nombre de façons de construire une surface de genre g à partir d'un polygone à $2g$ côtés, ont calculé la caractéristique d'Euler-Poincaré de l'espace $\mathcal{M}_g$; A. Zorich [Zor02] a montré que le comptage des surfaces à petits carreaux donne un moyen d'obtenir le volume des composantes connexes de strates ; ce travail a été réalisé ensuite par A. Eskin, A. Okounkov et R. Pandharipande dans [EO01] et [EOP08].

des sommets du graphe (les sommets sont des *arbres de relations*) a été réalisé par J. Cassaigne, S. Ferenczi et L. Zamboni [CFZ].

Soit $S_d^o \subset S_d$ l'ensemble des permutations irréductibles de S_d. Si on note $p(d) = |S_d^o|$ le nombre de permutations irréductibles de S_d alors on a la formule[7] :

$$p(d) = d! - \sum_{k=1}^{d-1} p(k)(d-k)!$$

Pour obtenir cette formule, il suffit de constater qu'une permutation réductible est la concaténation, de manière unique, d'une permutation irréductible de longueur k plus grande que 1 (le terme $p(k)$) et d'une autre permutation de longueur $d-k$ (le terme $(d-k)!$). Les premiers termes de cette suite sont donnés dans le tableau ci-dessous.

d	2	3	4	5	6	7	8	9	10	11
$p(d)$	1	3	13	71	461	3447	29093	273343	2829325	31998903
$p(d)/d!$	0.5	0.5	0.542	0.592	0.640	0.684	0.722	0.753	0.780	0.802

L'ensemble des classes de Rauzy forme une partition de S_d^o. Par exemple, les 71 permutations irréductibles de longueur 5 se partagent en 4 classes de Rauzy :

- la classe de (5 4 3 2 1) associée à la strate $\mathcal{H}(1;1) \simeq \mathcal{H}(1,1)$ qui contient 15 permutations ;
- la classe de (5 3 4 2 1) associée à la strate $\mathcal{H}(2;0)$ qui contient 35 permutations ;
- la classe de (5 3 2 4 1) associée à la strate $\mathcal{H}(0;2)$ qui contient 11 permutations ;
- la classe de (5 2 3 4 1) associée à la strate $\mathcal{H}(0;0,0) \simeq \mathcal{H}(0,0,0)$ qui contient 10 permutations.

Notre stratégie pour obtenir le comptage des classes de Rauzy se décompose en deux étapes : la première consiste à compter les permutations standards et la seconde à passer du comptage pour les permutations standards à celui pour toutes les permutations.

Une permutation $\pi \in S_d$ (resp. permutation étiquetée (π_t, π_b)) est dite *standard* si $\pi(1) = d$ et $\pi(d) = 1$ (resp. $\pi_t^{-1}(1) = \pi_b^{-1}(d)$ et $\pi_b^{-1}(1) = \pi_t^{-1}(d)$). Leur importance est en particulier due au fait suivant :

Chaque classe de Rauzy contient au moins une permutation standard.

Les permutations standards jouent un rôle important dans la classification des composantes connexes[8]. De plus elles ont toujours une position centrale dans les diagrammes de Rauzy (voir la position de $\binom{A\,B\,C\,D}{D\,C\,B\,A}$ dans le diagramme 2.5).

Dans la première partie de l'annexe C, nous démontrons que compter les permutations standards de chaque strate revient à compter le nombre de solutions d'une équation dans le groupe symétrique. On remarquera en particulier que le terme $z_\kappa = \prod_{i=1}^{s} (\kappa_i + 1)^{e_i} e_i!$ dans les théorèmes ci-dessous est le cardinal d'un centralisateur dans S_d. Suite aux travaux de G. Boccara [Boc80] et A. Goupil et G. Schaeffer [GS98] nous obtenons une formule explicite pour ces nombres. Ce comptage des permutations standards fait intervenir les méthodes de chirurgie de [KZ03] et [EMZ03] permettant

[7] Cette formule permet d'obtenir une asymptotique précise du nombre de permutations irréductibles : $p(d)$ est équivalent à $d!$ (voir [Com72]).

[8] Ces permutations ont une interprétation plus géométrique. Quitte à modifier légèrement la définition de suspension, on associe à une permutation standard une suspension faite d'un cylindre : on dit que c'est une *surface de translation Jenkins-Strebel*. Voir en particulier [Zor08].

de relier les composantes connexes entre elles. Combinatoirement, nous analysons les opérations qui consistent à enlever certains intervalles de la permutation. Pour cela, nous introduisons des strates avec un marquage. Une *sépartrice horizontale* d'une surface S est une demi-droite horizontale qui part d'une singularité. L'intervalle que constitue l'échange d'intervalles focalise deux singularités coniques qui correspondent à ses extrémités. Pour encoder ce marquage, nous définissons deux familles de strates. La strate $\mathcal{H}(m|a;\kappa')$ de l'ensemble des classes d'équivalence de surfaces de translation dans la strate $\mathcal{H}((m)\cup\kappa')$ dont on a marqué une séparatrice entrante et une séparatrice sortante sur le même zéro et dont l'angle (pour la métrique plate) entre ces deux séparatrices est $(2a+1)\pi$. La strate $\mathcal{H}(m_l \odot m_r;\kappa')$ désigne l'ensemble des classes d'équivalence de surfaces de translation dans la strate $\mathcal{H}((m_l,m_r)\cup\kappa')$ dont on a marqué une séparatrice entrante et une séparatrice sortante sur deux zéros distincts de degré respectivement m_l et m_r.

Soit π une permutation irréductible. On associe à toute suspension $S(\pi,\zeta)$ de π la strate marquée de la forme $\mathcal{H}(m|a;\kappa')$ ou $\mathcal{H}(m_l\odot m_r;\kappa')$ en considérant les séparatrices sortante et entrante sur $S(\pi,\zeta)$ associée au côté gauche et droit de π. On appellera *classe de Rauzy* associée à $\mathcal{H}(m_l\odot m_r;\kappa')$ (resp. $\mathcal{H}(m|a;\kappa')$ l'ensemble des permutations π de la classe de Rauzy étendue associée $\mathcal{H}((m_l,m_r)\cup\kappa')$ (resp. $\mathcal{H}(m|a);\kappa')$) dont le marquage induit par les extrémités de π est $m_l\odot m_r$ (resp. $m|a$). Nous donnons la liste des marquages des permutations de la strate $\mathcal{H}(1,1)$ dans la figure 2.7.

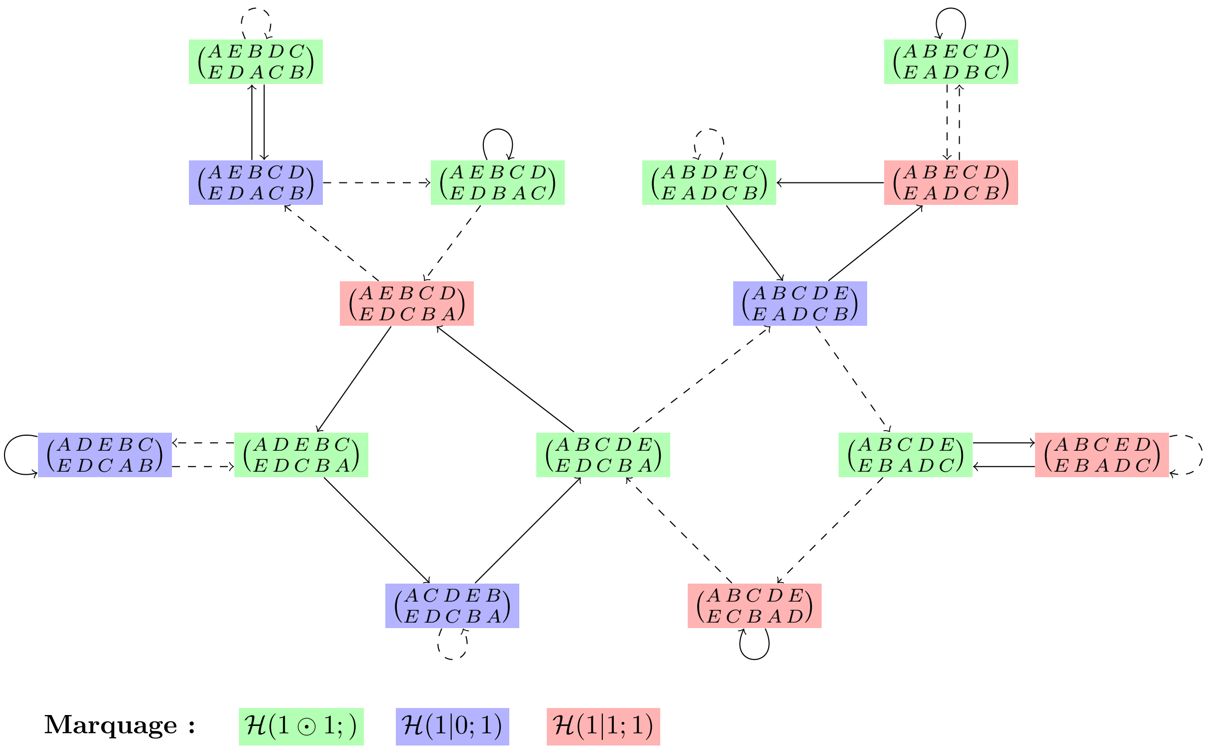


FIGURE 2.7 – Le diagramme de Rauzy associé à la strate $\mathcal{H}(1,1)$ contient 15 permutations. Il y a trois marquages $1\odot 1$ (7 permutations), $1|0$ (4 permutations) et $1|1$ (4 permutations).

Si $\kappa=(\kappa_1,\kappa_2,\ldots,\kappa_s)$ est une partition d'un entier on note $s(\kappa)=\kappa_1+\ldots+\kappa_s$ sa *somme* et $l(\kappa)=n$ sa *longueur*. On note également $z_{\kappa'}$ le cardinal du centralisateur de la partition $(\kappa_1+1,\kappa_2+1,\ldots,\kappa_s+1)$. Si e_i est le nombre d'occurrences de i dans cette partition alors : $z_{\kappa'}=\prod(\kappa_i+1)^{e_i}\ e_i!$. Nous obtenons en particulier :

I*l n'y a qu'une permutation standard dans les strates hyperelliptiques* $\mathcal{H}^{hyp}(2g-2)$ *(resp.* $\mathcal{H}^{hyp}(g-1,g-1)$*) qui, en notant* $d=2g$ *(resp.* $d=2g+1$*), est :*

$$\begin{pmatrix} 1 & 2 & \dots & d \\ d & d-1 & \dots & 1 \end{pmatrix}.$$

Pour la classe de Rauzy associée à la strate $\mathcal{H}_g(m|a;\kappa')$ *(resp.* $\mathcal{H}_g(m_l\odot m_r;\kappa')$*), notons* p *la partition* $(a,\kappa'_1+1,\dots,\kappa'_s+1)$ *(resp.* $(\kappa'_1+1,\dots,\kappa'_s+1)$*) et* $d=2g-2+s-1$*. Alors le nombre de permutations standards dans cette classe de Rauzy est :*

$$\frac{2(d-1)!}{(d+1)z_{\kappa'}}\sum_{q\subset p}\left((-1)^{s(q)+l(q)}\binom{d}{s(q)}^{-1}\right),$$

où la somme se fait sur toutes les sous-partitions q *de* p*.*
De plus, si κ *ne contient que des nombres pairs, alors la différence entre le nombre de permutations standards associées à* $\mathcal{H}^{odd}(a|m;\kappa)$ *et* $\mathcal{H}^{even}(a|m;\kappa')$ *est* 0 *si* $a\equiv 0 \mod 2$ *et sinon, en notant* $d=2g-2+s-1$*, est :*

$$\frac{(d-1)!}{2^{g-1}z_{\kappa'}}.$$

La différence entre le nombre de permutations standards associées à $\mathcal{H}^{odd}(m_l\odot m_r;\kappa')$ *et* $\mathcal{H}^{even}(m_l\odot m_r;\kappa')$ *est* $\frac{(d-1)!}{2^{g-1}z_{\kappa'}}$*.*

En particulier, nous obtenons des formules pour le nombre de permutations standards dans le cas particulier de la strate minimale $\mathcal{H}(2g-2)$.

L*a classe de Rauzy associée à* $\mathcal{H}_g^{hyp}(2g-2)$ *possède une seule permutation standard.*
Si g *est congru à* 1 *ou* 2 *modulo* 4 *alors les nombres de permutations standards de respectivement* $\mathcal{H}^{odd}(2g-2)$ *et* $\mathcal{H}^{even}(2g-2)$ *sont :*

$$(2g-2)!\left(\frac{1}{g}+\frac{1}{2^{g-1}}\right)-1 \qquad et \qquad (2g-2)!\left(\frac{1}{g}-\frac{1}{2^{g-1}}\right).$$

Si g *est congru à* 0 *ou* 3 *modulo* 4 *alors les nombres de permutations standards de respectivement* $\mathcal{H}^{odd}(2g-2)$ *et* $\mathcal{H}^{even}(2g-2)$ *sont :*

$$(2g-2)!\left(\frac{1}{g}+\frac{1}{2^{g-1}}\right) \qquad et \qquad (2g-2)!\left(\frac{1}{g}-\frac{1}{2^{g-1}}\right)-1.$$

Pour la *strate principale*, nous obtenons :

L*e nombre de permutations standards dans la classe de Rauzy associée à* $\mathcal{H}(1^{2k})$ *est :*

$$\frac{(4k-1)!}{(2k+1)2^{2k-1}(2k-1)!}$$

Passons à la seconde étape du comptage des permutations dans les classes de Rauzy. Nous analysons l'opération qui consiste à enlever les « bouts » d'une permutation standard[9]. Précisément, à une permutation standard π de S_{d+2} on associe la permutation $\tilde{\pi}$ dans S_d définie par $\tilde{\pi}(i)=\pi(i+1)-1$ pour $i=1,\dots,d$. Cette opération donne une bijection combinatoire triviale entre

[9]Cette opération joue un rôle essentiel dans la classification des composantes connexes de strates [KZ03] et [Lan08].

les permutations standards de S_{d+2} et les permutations (pas nécessairement irréductibles) de S_d. Prenons l'exemple, du diagramme de Rauzy R associé à la strate $\mathcal{H}^{odd}(4)$ qui contient 7 permutations standards. En appliquant l'opération $\pi \mapsto \tilde{\pi}$, six d'entre elles arrivent dans la strate $\mathcal{H}(2)$:

$$(4\,2\,1\,3), \quad (3\,1\,4\,2), \quad (2\,4\,3\,1), \quad (4\,1\,3\,2), \quad (3\,2\,4\,1), \quad (2\,4\,1\,3),$$

tandis que la dernière est réductible est correspond à la concaténation de deux permutations de $\mathcal{H}(0)$:

$$(2\,1\,4\,3) = (2\,1) \cdot (2\,1).$$

En introduisant des suspensions pour les permutations réductibles, nous parvenons à les classer en composantes connexes. Une technique d'inclusion-exclusion, similaire à celle utilisée pour compter les permutations irréductibles, permet d'obtenir une formule pour la cardinalité de chaque classe de Rauzy. Cette dernière ne fait intervenir que la combinatoire des partitions d'entiers et les nombres de permutations standards obtenus dans la première étape du comptage.

En outre, C. Boissy [Boib] démontre que l'étiquetage des permutations revient à faire un marquage des surfaces de translation en donnant un nom à chaque séparatrice sortante. Notons $\mathcal{H}_g^{lab}$ ces espaces des modules des surfaces de translation marquées. Le degré du revêtement $\mathcal{H}^{lab}(m;\kappa') \to \mathcal{H}(m;\kappa')$ est égal, pour chaque composante connexe, au rapport entre le cardinal d'une classe de Rauzy étiquetée et réduite. Le théorème suivant de l'article [Boib] explicite ce degré :

> S*oit* m *un entier positif ou nul et* $\kappa' = (\kappa_1, \ldots, \kappa_s)$ *une partition avec* $2g-2 = m + \sum \kappa_i$*. On note* $\kappa = (m) \uplus \kappa'$*. Si la strate* $\mathcal{H}(\kappa)$ *contient une composante hyperelliptique alors le degré du revêtement* $\mathcal{H}_g^{lab,hyp}(m;\kappa') \to \mathcal{H}^{hyp}(m;\kappa')$ *est* 1*. Pour toute composante* $\mathcal{C}(m;\kappa')$ *de la strate* $\mathcal{H}(m;\kappa')$ *différente de la composante hyperelliptique, le degré du revêtement* $\mathcal{C}^{lab} \to \mathcal{C}$ *est* $\varepsilon z_{\kappa'}$ *où :*
>
> $$\varepsilon = \varepsilon(\kappa) = \begin{cases} 1/2 & \textit{si un des } \kappa_i \textit{ est pair} \\ 1 & \textit{sinon} \end{cases}.$$

Joints à ceux de C. Boissy, nos résultats donnent des formules pour le cardinal de toutes les classes de Rauzy. À notre connaissance, il n'existe pas de telles formules pour les classes de Rauzy de différentielles quadratiques.

Comptage pour les composantes de strates pour $g \leq 5$

composante	perm. std.	card. cl. R.
g=1		
$\mathcal{H}(0)$	1	1
g=2		
$\mathcal{H}(2)$	1	7
$\mathcal{H}(1,1)$	1	15
g=3		
$\mathcal{H}^{hyp}(4)$	1	31
$\mathcal{H}^{hyp}(2,2)$	1	63
$\mathcal{H}^{odd}(4)$	7	134
$\mathcal{H}(3,1)$	24	770
$\mathcal{H}^{odd}(2,2)$	11	294
$\mathcal{H}(2,1,1)$	49	2177
$\mathcal{H}(1^4)$	21	1255
g=4		
$\mathcal{H}^{hyp}(6)$	1	127
$\mathcal{H}^{hyp}(3,3)$	1	255
$\mathcal{H}^{odd}(6)$	135	5209
$\mathcal{H}^{even}(6)$	44	2327
$\mathcal{H}(5,1)$	720	41574
$\mathcal{H}^{odd}(4,2)$	472	23506
$\mathcal{H}^{even}(4,2)$	136	10568
$\mathcal{H}^{nonhyp}(3,3)$	275	15568
$\mathcal{H}(4,1,1)$	1728	128492
$\mathcal{H}(3,2,1)$	2952	217349
$\mathcal{H}^{odd}(2^3)$	372	23167
$\mathcal{H}^{even}(2^3)$	92	9876
$\mathcal{H}(3,1^3)$	3240	301586
$\mathcal{H}(2,2,1,1)$	4440	408533
$\mathcal{H}(2,1^4)$	5445	617401
$\mathcal{H}(1^6)$	1485	202571

g=5

$\mathcal{H}^{hyp}(8)$	1	511
$\mathcal{H}^{hyp}(4,4)$	1	1023
$\mathcal{H}^{odd}(8)$	5291	352697
$\mathcal{H}^{even}(8)$	2772	233285
$\mathcal{H}(7,1)$	40320	3697874
$\mathcal{H}^{odd}(6,2)$	21240	1742192
$\mathcal{H}^{even}(6,2)$	10440	1120946
$\mathcal{H}(5,3)$	27360	2494234
$\mathcal{H}^{odd}(4,4)$	8891	729495
$\mathcal{H}^{even}(4,4)$	4356	469943
$\mathcal{H}(4,3,1)$	163152	18245942
$\mathcal{H}^{odd}(4,2,2)$	51348	5072573
$\mathcal{H}^{even}(4,2,2)$	23628	3174918
$\mathcal{H}(4,2,1,1)$	598752	80343780
$\mathcal{H}(4,1^4)$	442728	70584695
$\mathcal{H}(3,3,2)$	69300	7692855
$H(3,3,1,1)$	279180	37568302
$H(3,2,2,1)$	506880	67631764
$H(3,2,1^3)$	1492920	237181716
$H(3,1^5)$	720720	134001474
$H^{odd}(2^4)$	27060	3163511
$\mathcal{H}^{even}(2^4)$	11660	1924730
$\mathcal{H}(2^3,1,1)$	674960	106542326
$\mathcal{H}(2,2,1^4)$	1621620	300296573
$\mathcal{H}(2,1^6)$	1126125	241202517
$\mathcal{H}(1^8)$	225225	55184875

Figure 2.8 – Nombre de permutations standards et nombre de permutations dans les classes de Rauzy étendues pour les strates en genre inférieur ou égal à 5.

Chapitre 3

Cocycle de Kontsevich-Zorich

Dans le chapitre précédent nous avons vu les actions du flot de Teichmüller et de l'induction de Rauzy-Veech sur les suspensions d'échanges d'intervalles. Ces deux opérations simultanées forment un processus de renormalisation que nous utilisons dans ce chapitre pour étudier des sommes de Birkhoff au-dessus des échanges d'intervalles. Les déviations de ces sommes de Birkhoff sont contrôlées par le comportement asymptotique du cocycle de Kontsevich-Zorich qui est le produit de matrices décrivant la suite de changement de bases effectuées lors de l'induction de Rauzy.

Cette partie est une étape essentielle de la preuve du théorème sur le taux de diffusion pour le vent dans les arbres.

3.1 Déviations des sommes de Birkhoff : un exemple

Nous étudions les sommes de Birkhoff au-dessus de l'échange d'intervalles introduit dans la section 2.1. Soit $T = T_{\pi,\lambda}$ l'échange d'intervalles avec $\pi = \binom{A\,B\,C\,D}{D\,C\,B\,A}$ et $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$ le vecteur propre de Perron-Frobenius (à droite) de la matrice :

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix}.$$

Comme nous l'avons vu dans la section 2.1, la matrice M est la matrice du cocycle de Kontsevich-Zorich de (π, λ) au temps 4.3902. Remarquons tout d'abord que les valeurs propres de M sont symétriques : $\theta_1 \simeq 4.3902$, $\theta_2 \simeq 1.8379$, $1/\theta_2 \simeq 0.5441$ et $1/\theta_1 \simeq 0.2278$. En particulier, θ_1 et θ_2 sont strictement supérieures à 1 tandis que θ_3 et θ_4 sont strictement inférieurs à 1.

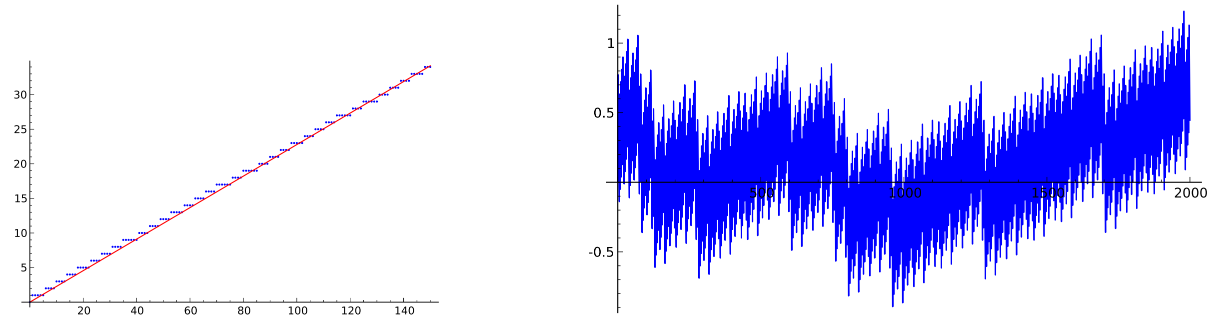
Soit χ_A la fonction caractéristique de l'intervalle portant l'étiquette A alors $S_N(T, f_A, x)$ compte combien de fois l'orbite de x jusqu'au temps N passe par l'intervalle I_A. Plus généralement, considérons une fonction $f = f_A\chi_A + f_B\chi_B + f_C\chi_C + f_D\chi_D$. Soit $S_N(T, x) = (N_A, N_B, N_C, N_D)$ le vecteur composé des nombres de fois que l'orbite de x jusqu'au temps N visite respectivement les intervalles I_A, I_B, I_C et I_D. La somme de Birkhoff $S_N(T, f, x)$ de f se réécrit comme un produit scalaire $S_N(T, f, x) = \langle f, S_N(T, x)\rangle$ où f est vu comme le vecteur (f_A, f_B, f_C, f_D).

La moyenne de la fonction f est $\lambda(f) = \langle f, \lambda\rangle$, en particulier $\lambda(\chi_A) = \lambda_A$. Dans la section 1.5, nous avons évoqué que $S_N(T, f, x)$ est en première approximation, de la taille de $N\lambda(f)$ (voir la partie gauche de la figure 3.1a). Plus précisément,

$$\frac{S_N(T, f, x)}{N} \xrightarrow[N\to\infty]{} \lambda(f).$$

Nous nous intéressons alors à la différence $S_N(T,f,x) - N\lambda(f)$. Si on note $\mathbf{1} = (1,1,1,1)$, cette différence se réécrit $\langle S_N(T,x), f - \lambda(f)\mathbf{1}\rangle$ (voir la partie droite de la figure 3.1b).

La différence $f - \lambda(f)\mathbf{1}$ est la projection du vecteur f sur l'espace des vecteurs de moyenne nulle (l'orthogonal de λ) parallèlement à la direction donnée par les fonctions constantes ($\mathbb{R}\mathbf{1}$). Notons $v_1 = \lambda$, v_2, v_3, v_4 les vecteurs propres à droites de la matrice M. Il existe une base de vecteurs propres à gauche v_1^*, v_2^*, v_3^*, v_4^* normalisée pour que $\langle v_i^*, v_j\rangle = \delta_{ij}$ où δ_{ij} est le symbole de Kronecker. L'espace $\mathrm{Vect}(v_2^*, v_3^*, v_4^*)$ coïncide avec l'orthogonal de $v_1 = \lambda$ et correspond à l'ensemble des fonctions de moyennes nulles. En particulier, le vecteur $f - \lambda(f)\mathbf{1}$ est combinaison linéaire de v_2^*, v_3^* et v_4^*.



(a) Les valeurs de la somme de Birkhoff $S_N(T,\chi_A,x)$ (points en bleu) et la droite $y = \lambda_A x$ (en rouge) pour $N \leq 100$.

(b) La différence $S_N(T,\chi_A,x) - N\lambda_A$ pour $N \leq 5000$.

FIGURE 3.1 – Déviations des sommes de Birkhoff de la fonction χ_A (fonction indicatrice de l'intervalle A) au-dessus de l'échange d'intervalles de la figure 2.1.

Quitte à remplacer, f par $f - \lambda(f)$ on peut supposer qu'elle est de moyenne nulle, autrement dit que f appartient à $\mathrm{Vect}(v_2^*, v_3^*, v_4^*)$. Nous utilisons maintenant la matrice M pour réécrire les sommes de Birkhoff de la fonction f. Soit x le point gauche du domaine de $T_{\pi,\lambda}$ et N la somme de la première colonne de M^n. La taille de N est approximativement θ_1^n ce qui nous permet de réécrire :

$$\begin{aligned}
\frac{\log|\langle f, S_N(T,x)\rangle|}{\log N} &\simeq \frac{\log|\langle f, M^n e\mathbf{1}\rangle|}{n\log(\theta_1)} \\
&= \frac{\log|\langle (M^n)^* f, e\mathbf{1}\rangle|}{n\log(\theta_1)} \\
&\simeq \frac{\log\|(M^n)^* f\|}{n\log(\theta_1)}
\end{aligned}$$

Nous ne justifions pas les deux erreurs d'approximations commises ci-dessus, mais nous insistons que nous les avons faite uniquement pour un point x et un entier N bien choisis. Cette dernière quantité est de l'ordre de $log(\theta_i)/\log(\theta_1)$ où $i = 2,3,4$ suivant que $f \in \mathrm{Vect}(v_2^*, v_3^*, v_4^*)$, $f \in \mathrm{Vect}(v_3^*, v_4^*)$ ou bien $f \in \mathrm{Vect}(v_4^*)$. On peut démontrer le résultat précis suivant :

N*otons* $\nu_2 = \log\theta_2/\log\theta_1$ *et* $V_1 = \mathbb{R}^4$, $V_2 = \text{Vect}(v_2^*, v_3^*, v_4^*)$, $V_3 = \text{Vect}(v_3^*, v_4^*)$. *Alors :*

- *pour tout* $f \in V_1 \backslash V_2$ *et tout point* x *:,*

$$\lim_{N\to\infty} \frac{S_N(f,T,x)}{N} = \lambda(f) \neq 0;$$

- *pour* $f \in V_2 \backslash V_3$ *et tout point* x*,*

$$\limsup_{N\to\infty} \frac{\log|S_N(f,T,x)|}{\log N} = \nu_2;$$

- *pour* $f \in V_2$*, il existe une constante* C *telle que :*

$$\|S_N(f,T,x)\| \leq C$$

Ce théorème précise ainsi le théorème de Birkhoff qui correspond au premier item. L'objectif de la partie suivante est de présenter une généralisation de ce théorème.

3.2 Déviations des sommes de Birkhoff : le cas général

Nous voulons étudier les déviations des sommes de Birkhoff pour un échange d'intervalles dont l'induction de Rauzy n'est pas nécessairement périodique. Dans la partie 1.5, nous avons évoqué le théorème de Birkhoff qui décrit l'asymptotique d'une somme prise le long de l'orbite d'une transformation. Il existe une version de ce théorème pour les matrices : le théorème d'Oseledets [Ose68]. Ce dernier donne une description asymptotique d'un produit de matrices qui, dans le cas des matrices transposées $\left(M^{(t)}(\pi,\lambda)\right)^*$ du cocycle de Kontsevich-Zorich, s'énonce :

S*oit* π *une permutation étiquetée irréductible sur l'alphabet* $\mathcal{A}$*. Notons* $\mathcal{C} \subset \mathcal{H}(\kappa)$ *la composante connexe de strate correspondant au digramme de Rauzy de* π*. Soit* μ *une mesure de probabilité sur* $\mathcal{C}$*, ergodique pour le flot de Teichmüller. Alors, il existe des réels* $\nu_1 > \nu_2 > \dots \nu_k > 0$ *tels que : pour* μ*-presque toute donnée de longueur* $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ *pour* π*, il existe un drapeau :*

$$\mathbb{R}^{\mathcal{A}} = V_1^u(\lambda) \supset V_2^u(\lambda) \supset \dots \supset V_k^u(\lambda) \supset V^c(\lambda) \supset V_k^s(\lambda) \supset \dots \supset V_1^s(\lambda) \supset \{0\}$$

et une suite $(t_l)_{\ell\geq 0}$ *de temps tendant vers l'infini*[1] *vérifiant :*

1. *pour tout* $f \in V_i^u(\lambda) \backslash V_{i+1}^u(\lambda)$ *:* $\lim_{\ell\to\infty} \dfrac{\log\left\|\left(M^{(t_\ell)}(\pi,\lambda)\right)^* f\right\|}{t_\ell} = \nu_i$ *;*

2. *pour tout* $f \in V^c(\lambda) \backslash V_k^s(\lambda)$ *:* $\lim_{\ell\to\infty} \dfrac{\log\left\|\left(M^{(t_\ell)}(\pi,\lambda)\right)^* f\right\|}{t_\ell} = 0.$

3. *pour tout* $f \in V_i^s(\lambda) \backslash V_{i-1}^s(\lambda)$ *:* $\lim_{\ell\to\infty} \dfrac{\log\left\|\left(M^{(t_\ell)}(\pi,\lambda)\right)^* f\right\|}{t_\ell} = -\nu_i$ *;*

[1] Toute suite de temps telle que $g_{t_l} \cdot S$ reste dans un sous-ensemble compact de $\mathcal{C}$ convient.

Les réels ν_i, $-\nu_i$ pour $i = 1, \ldots, k$ et éventuellement 0 si $V^c \neq \emptyset$ sont les *exposants de Lyapunov du cocycle de Kontsevich-Zorich* [2]. Les dimensions m_i des espaces V_i^u et V_i^s ne dépendent pas de λ. Ce sont les *multiplicités* des exposants de Lyapunov. Ces exposants et leurs multiplicités généralisent la notion de valeurs propres d'une matrice. L'espace $V^u = \bigoplus V_i^u$ (resp. $V^s = \bigoplus V_i^s$ et V^c) s'appelle l'*espace instable* (resp. *espace stable* et *espace central*) du cocycle de Kontsevich-Zorich et généralisent les espaces propres. Il faut prendre garde que dans ce cadre, même si les exposants de Lyapunov ne dépendent pas de la valeur λ, les espaces V_i^u, V_i^s et V^c en dépendent.

Par définition du cocycle de Kontsevich-Zorich, $\nu_1 = 1$. D'autre part, W. Veech [Vee86] démontre que, pour toute composante connexe de strate munie de la mesure de Lebesgue, on a $\nu_1 > \nu_2$. Ce résultat sera étendu à toute mesure g_t-ergodique par G. Forni [For02, For11]. M. Kontsevich et A. Zorich ont conjecturé que, pour les mesures de Lebesgue sur les composantes connexes de strates, le spectre était toujours *simple* : les multiplicités vérifient $m_1 = m_2 = \ldots = m_k = 1$ et $m = 0$. A. Avila et M. Viana [AV07b] démontrent cette conjecture et, contrairement à l'approche géométrique de G. Forni [For02, For11], leur preuve utilise la version discrète du cocycle de Kontsevich-Zorich donnée par l'induction de Rauzy. La preuve de ce théorème repose sur un critère de simplicité des exposants de Lyapunov des mêmes auteurs [AV07a] qui peut s'appliquer à d'autres cas. Par exemple, C. Matheus, M. Möller et J.-C. Yoccoz [MMY] développent un critère suffisant de simplicité du spectre de Lyapunov du cocycle de Kontsevich-Zorich des surfaces à petits carreaux. Ce critère nous a permis, dans un travail en collaboration avec C. Matheus [DM], de fournir un contrexemple à la réciproque du théorème 2 de l'article [For11] de G. Forni.

Le théorème d'Oseledets généralise dans un cadre dynamique la décomposition en valeur propre/vecteur propre d'une matrice. Nous énonçons maintenant comment ce théorème suffit à généraliser le résultat sur les déviations des sommes de Birkhoff dans le cas où l'induction de Rauzy n'est plus périodique. M. Kontsevich et A. Zorich [Kon97] ont conjecturé que le second exposant de Lyapunov était responsable des déviations des moyennes ergodiques. A. Zorich donne une démonstration complète pour les échanges d'intervalles et les surfaces de translation [Zor96, Zor97, Zor99] (les résultats sont génériques relativement à la mesure de Lebesgue). G. Forni complète les résultats de A. Zorich en faisant un lien avec des obstructions d'une équation cohomologique pour les distributions [For97, For02]. Dans l'annexe B, nous étendons le théorème de [Zor99] et une partie du théorème de [For02] afin de démontrer le résultat de diffusion pour le vent dans les arbres.

[2] La symétrie de ces exposants est due au fait que le cocycle de Kontsevich-Zorich est symplectique. Ceci entraîne également une orthogonalité au niveau des espaces V_i^u, V_i^s et V^c (voir la section 3.3).

S*oit* π *une permutation irréductible sur l'alphabet* $\mathcal{A}$ *et* $\mathcal{C}$ *la composante connexe de strate associée au digramme de Rauzy de* π. *Soit* μ *une mesure de probabilité sur* $\mathcal{C}$ *ergodique pour le flot de Teichmüller et* $(\nu_i)_{i=1,\dots,k}$ *les exposants de Lyapunov du cocycle de Kontsevich-Zorich pour cette mesure. Pour une donnée* λ *de longueurs Oseledets génériques, on note :*

$$\mathbb{R}^{\mathcal{A}} = V_1^u(\lambda) \supset V_2^u(\lambda) \supset \ldots \supset V_k^u(\lambda) \supset V^c(\lambda) \supset V_k^s(\lambda) \supset \ldots \supset V_1^s(\lambda) \supset \{0\}$$

le drapeau d'Oseledets du cocycle de Kontsevich-Zorich et $T = T_{\pi,\lambda} : I \to I$ *l'échange d'intervalles de données* (π, λ). *Alors, pour tout point* $x \in I$ *:*

1. *dans l'espace instable, la croissance des sommes de Birkhoff est polynomiale :*

$$\text{pour } i = 1, \ldots, k \text{ et } f \in V_i^u(\lambda) \backslash V_{i+1}^u(\lambda), \quad \limsup_{T\to\infty} \frac{\log |\langle S_N(T,x), f\rangle|}{\log N} = \nu_i;$$

2. *dans l'espace central, la croissance est sous-exponentielle :*

$$\text{pour } f \in V^c(\lambda) \backslash V^s(\lambda), \quad \limsup_{T\to\infty} \frac{\log |\langle S_N(T,x), f\rangle|}{\log N} = 0;$$

3. *dans l'espace stable, la croissance est bornée. Il existe une constante* C *(dépendant de* λ*) telle que :*

$$\text{pour } f \in V^s(\lambda) \backslash 0, \quad \forall T \geq 0, \ |\langle S_N(T,x), f\rangle| \leq C.$$

Les exposants de Lyapunov donnent donc un moyen de contrôle sur les déviations des sommes de Birkhoff. Pour appliquer le théorème ci-dessus au vent dans les arbres, il reste deux points à aborder :

- localiser la position du cocycle du vent dans les arbres dans le drapeau d'Oseledets et faire le calcul de l'exposant de Lyapunov associé ;
- montrer que les surfaces $\mathrm{X}(a, b)$ sont Oseledets génériques afin de pouvoir appliquer le théorème ci-dessus.

3.3 Un point de vue plus géométrique sur le cocycle de Kontsevich-Zorich

Nous avons introduit le cocycle de Kontsevich-Zorich en utilisant les échanges d'intervalles. Cette description permet d'écrire très explicitement le processus de renormalisation. Dans cette section, nous suggérons une approche plus géométrique en utilisant les groupes d'homologie et de cohomologie des surfaces de translation. Nous n'utilisons que des objets élémentaires de topologie algébrique et expliquons la symétrie des exposants de Lyapunov du cocycle de Kontsevich-Zorich.

Définissons les groupes d'homologie et de cohomologie[3]. Soit S une surface de translation de genre g dans la strate $\mathcal{H}(\kappa_1, \kappa_2, \ldots, \kappa_s)$. On note $\Sigma = \{P_1, P_2, \ldots, P_s\} \subset S$ l'ensemble des singularités coniques de la surface S. Le premier groupe d'homologie $H_1(S; \mathbb{R})$ de S est un $\mathbb{R}$-espace vectoriel dont les éléments sont des sommes formelles de courbes fermées. Une courbe γ est nulle dans $H_1(S; \mathbb{R})$ si et seulement si elle borde un disque. L'espace $H_1(S \backslash \Sigma; \mathbb{R})$ est définie de la même

[3]Pour une référence concernant tous les concepts de géométrie algébrique, nous renvoyons à l'ouvrage [Hat02].

façon mais pour la surface S dont on a enlevé les singularités. Ce dernier est engendré par les courbes évitant les singularités de S.

> S*oit S une surface de translation genre g dont les singularités coniques sont $\Sigma = \{P_1, P_2, \ldots, P_s\}$. Alors :*
>
> $$\dim H_1(S;\mathbb{R}) = 2g \qquad et \qquad \dim H_1(S\backslash\Sigma;\mathbb{R}) = 2g + s - 1.$$
>
> *La surjection naturelle $H_1(S\backslash\Sigma;\mathbb{R}) \to H_1(S;\mathbb{R})$ a pour noyau E_0 l'espace vectoriel de dimension $s-1$ engendré par les s petits cercles autour des singularités.*

Soit $S = S(\pi,\zeta)$ une suspension d'une permutation irréductible $\pi = (\pi_t, \pi_b)$ sur l'alphabet $\mathcal{A}$ (voir la section 2.1). Une base naturelle de $H_1(S\backslash\Sigma;\mathbb{R})$ est donnée par les courbes e_α joignant le côté ζ_α sur L_b au vecteur ζ_α sur L_t où, comme dans la section 2.1, L_b et L_t sont les lignes brisées obtenues en concaténant les vecteurs ζ_α (voir la figure 3.2). Du point de vue des rectangles cousus, la courbe e_α traverse une fois le rectangle α et ne passe pas dans les autres. Par construction, ces vecteurs joignent deux singularités coniques de la surface S. L'espace vectoriel $H_1(S\backslash\Sigma;\mathbb{R})$ s'identifie donc de manière naturel avec $\mathbb{R}^{\mathcal{A}}$ et c'est sur ce dernier que nous avions fait agir les matrices $M^{(t)} = M_1\, M_2\, \ldots\, M_n$ du cocycle de Kontsevich-Zorich. Mais il faut noter qu'en écrivant $\mathbb{R}^{\mathcal{A}}$ pour une suspension, nous faisons le choix d'une base privilégiée de $H_1(S\backslash\Sigma;\mathbb{R})$.

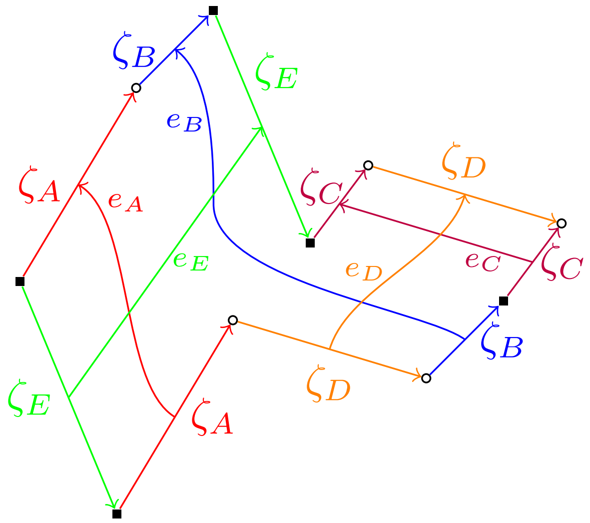


FIGURE 3.2 – Suspension de la permutation $\pi = \binom{A\,B\,E\,C\,D}{E\,A\,D\,B\,C}$ dans la strate $\mathcal{H}(1,1)$. Les deux singularités coniques d'angle 3π sont représentées par un carré plein et un cercle creux. Les courbes joignant les côtés $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ forment une base de $H_1(S\backslash\Sigma;\mathbb{R})$ et les côtés $\{\zeta_\alpha\}_{\alpha\in\mathcal{A}}$ forment sa base duale dans $H_1(S,\Sigma;\mathbb{R})$.

Dans la section précédente, nous avons vu que, pour les déviations des sommes de Birkhoff, le rôle principal était joué par la transposée des matrices $M^{(t)} = M_1\, M_2\, \ldots\, M_n$ définie par l'induction de Rauzy. Ces dernières agissent sur le dual de $\mathbb{R}^{\mathcal{A}}$. D'un point de vue géométrique, il s'agit du *groupe de cohomologie* $H^1(S\backslash\Sigma;\mathbb{R}) = H_1(S\backslash\Sigma;\mathbb{R})^*$. Cette dualité peut se voir plus directement sur la surface. Le premier groupe d'homologie relative $H_1(S,\Sigma;\mathbb{R})$ est le $\mathbb{R}$-espace vectoriel engendré par les courbes fermées et les courbes dont les deux bouts sont dans Σ. On a une inclusion $H_1(S;\mathbb{R}) \to H_1(S,\Sigma;\mathbb{R})$ car tout cycle est en particulier un cycle relatif. Si $S = S(\pi,\zeta)$ est une suspension, une base naturelle de $H_1(S,\Sigma;\mathbb{R})$ est donné par les vecteurs ζ_α qui forment les côtés L_t et L_b. Les courbes e_α et ζ_α

vérifient la propriété particulière que e_α intersecte ζ_α exactement une fois et n'intersecte pas ζ_β pour $\beta \neq \alpha$. C'est l'*intersection* qui permet de définir cette dualité.

La *forme d'intersection* Ω_S d'une surface S est l'application bilinéaire antisymétrique définie de la manière suivante. L'intersection de deux courbes $\Omega_S(\gamma_1, \gamma_2)$ est le nombre d'intersection, compté avec multiplicités, de γ_1 avec γ_2. En étendant par linéarité, Ω_S définit une forme bilinéaire antisymétrique sur $H_1(S;\mathbb{R})$. Comme il existe un morphisme surjectif $H_1(S\backslash\Sigma;\mathbb{R}) \to H_1(S;\mathbb{R})$ la forme d'intersection est également définie sur $H_1(S\backslash\mathbb{R})$ et plus généralement sur $H_1(S\backslash\Sigma;\mathbb{R}) \times H_1(S,\Sigma;\mathbb{R})$.

> L*a forme* Ω_S *est bien définie et non dégénérée sur* $H_1(S;\mathbb{R}) \times H_1(S;\mathbb{R})$ *et* $H_1(S\backslash\Sigma;\mathbb{R}) \times H_1(S,\Sigma;\mathbb{R})$. *Sur* $H_1(S\backslash\Sigma;\mathbb{R}) \times H_1(S\backslash\Sigma;\mathbb{R})$ *elle est de rang* $2g$ *et son noyau est* E_0.

Le résultat ci-dessus montre qu'il existe un isomorphisme canonique $H^1(S\backslash\Sigma;\mathbb{R}) \simeq H_1(S,\Sigma;\mathbb{R})$. Dans le cas d'une suspension, les vecteurs e_α et ζ_α sont duaux pour la forme d'intersection.

Pour une suspension $S = S(\pi,\zeta)$ la forme d'intersection sur les vecteurs e_α se calcule simplement à partir de π :

$$\Omega_{\alpha,\beta} = \begin{cases} 1 & \text{si } \pi_t(\alpha) < \pi_t(\beta) \text{ et } \pi_b(\alpha) > \pi_b(\beta), \\ -1 & \text{si } \pi_t(\alpha) > \pi_t(\beta) \text{ et } \pi_b(\alpha) < \pi_b(\beta), \\ 0 & \text{sinon} \end{cases}$$

La matrice Ω_π enregistre les croisements de la permutation π. Par exemple, pour la permutation $\pi = \begin{pmatrix} A\,B\,E\,C\,D \\ E\,A\,D\,B\,C \end{pmatrix}$ de la figure 3.2 cette matrice est :

$$\Omega_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Le vecteur $e_A - e_B + e_C \in H_1(S\backslash\Sigma;\mathbb{R})$ engendre le noyau de Ω_π et correspond à une courbe qui fait le tour d'un des zéros de la suspension de la figure 3.2.

La transposée $\left(M^{(t)}\right)^*$ du cocycle de Kontsevich-Zorich agissant sur $H_1(S,\Sigma;\mathbb{R}) \simeq \langle\{\zeta_\alpha\}\rangle$ préserve la forme d'intersection Ω_S ainsi que la décomposition $E_0 \oplus H^1(S;\mathbb{R})$. On peut également montrer que sur la partie E_0, elle agit de manière bornée et qu'en particulier les exposants de Lyapunov associés sont nuls. Ces deux remarques impliquent la forme particulière des exposants de Lyapunov :

> S*oit* π *une permutation irréductible et* s *le nombre de singularités coniques d'une suspension de* π. *Soit* $\mathcal{C}$ *la composante de strates de* π *et* μ *une mesure ergodique sur* $\mathcal{C}$. *Parmi les exposants du cocycle de Kontsevich-Zorich pour la mesure* μ, *il y a au moins* $s-1$ *zéros qui correspondent à l'action de* $\left(M^{(t)}\right)^*$ *sur la partie relative de la cohomologie* E_0. *Sur la partie absolue,* $H^1(S;\mathbb{R})$, *le cocycle est symplectique (il préserve la forme d'intersection non dégénérée* Ω_S*) et les* $2g$ *exposants sont regroupés par couples de réels opposés* $(\nu_i, -\nu_i)$.

Dans toute la suite on notera souvent $1 = \nu_1 > \nu_2 \geq \nu_3 \geq \ldots \geq \nu_g \geq 0$ les exposants positifs du cocycle de Kontsevich-Zorich.

Chapitre 4

Action de $\mathrm{SL}(2,\mathbb{R})$ sur les strates de surfaces de translation

Dans le chapitre précédent nous avons étudié l'action du flot de Teichmüller sur les strates $\mathcal{H}(\kappa)$ de surfaces de translation. Ce flot s'identifie à une action linéaire du sous-groupe à un paramètre $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ et permet d'étudier les phénomènes asymptotiques du flot linéaire des surfaces de translation. Dans ce chapitre, nous considérons l'extension de cette action à $\mathrm{SL}(2,\mathbb{R})$. Les propriétés dynamiques de cette dernière sont beaucoup plus rigides. Par exemple, dans le cas de la strate $\mathcal{H}(2)$, il est impossible de donner une classification raisonnable des mesures ergodiques pour le flot de Teichmüller alors qu'il existe une classification simple des mesures $\mathrm{SL}(2,\mathbb{R})$-invariantes. C'est en utilisant cette classification ainsi que des éléments de géométrie algébrique que nous parvenons à calculer la valeur explicite 2/3 de l'exposant de Lyapunov du cocycle de Kontsevich-Zorich qui contrôle la diffusion du vent dans les arbres.

4.1 Action de $\mathrm{SL}(2,\mathbb{R})$ sur une surface de translation

Nous pouvons étendre le flot de Teichmüller à une action de $\mathrm{SL}(2,\mathbb{R})$: une matrice de $\mathrm{SL}(2,\mathbb{R})$ agit linéairement sur les coordonnées des polygones définissant une surface de translation. Considérerons les deux sous-groupes à un paramètre suivants :

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

L'action de r_θ consiste à faire tourner les surfaces de translation. Autrement dit, pour une surface de translation S, le flot linéaire de $r_\theta \cdot S$ est le flot linéaire de S dans la direction $-\theta$. L'action des matrices u_s s'appelle le *flot unipotent* (voir la figure 4.1). Nous notons $A = \{g_t\}_{t\in\mathbb{R}}$ le sous-groupe diagonal, $K = \{r_\theta\}_{\theta\in S^1}$ l'ensemble des matrices de rotation et $N = \{u_s\}_{s\in\mathbb{R}}$ l'ensemble des matrices unipotentes. Ces trois sous-groupes décomposent $\mathrm{SL}(2,\mathbb{R})$: la *décomposition d'Iwasawa* de $\mathrm{SL}(2,\mathbb{R})$ est la décomposition unique de chaque matrice de $\mathrm{SL}(2,\mathbb{R})$ en un produit $k\ a\ n$ avec $k \in K$, $a \in A$ et $n \in N$. Notons que, mis à part le flot de Teichmüller, les rotations ne jouent pas un grand rôle (K est un groupe compact) et la dynamique du flot unipotent reste relativement incomprise.

Tout comme le flot de Teichmüller, l'action de $\mathrm{SL}(2,\mathbb{R})$ préserve l'aire des surfaces et les degrés des singularités. En particulier, il agit sur les strates. Comme $\mathrm{SL}(2,\mathbb{R})$ est connexe, il préserve également les composantes connexes de strates. Les échanges d'intervalles, fort util pour comprendre la dynamique du flot de Teichmüller, ne sont d'aucun secours pour comprendre l'action de $\mathrm{SL}(2,\mathbb{R})$.

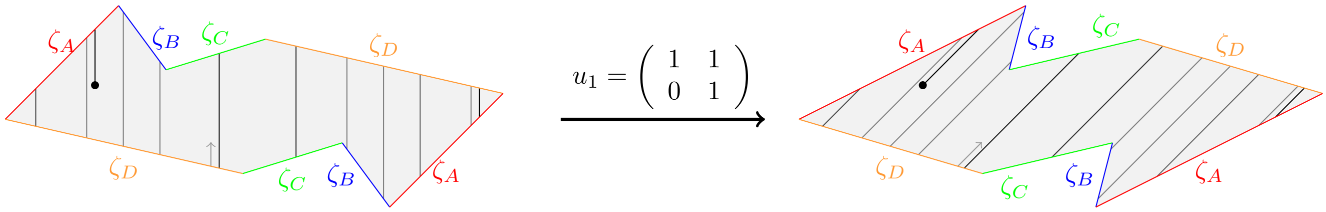


FIGURE 4.1 – Action du flot unipotent sur une surface de translation. La surface de gauche est la même que dans la figure 2.1 page 12.

Nous pouvons voir sur la figure 4.1 que l'image de la suspension par la matrice u_1 n'est plus une suspension.

Revenons à la surface $\mathrm{X}(a,b)$ construite pour étudier le vent dans les arbres (voir la partie 1.3). Nous cherchons des propriétés génériques des flots linéaires de $\mathrm{X}(a,b)$ relativement à la direction du flot ; autrement dit, aux flots linéaires verticaux de la famille de surfaces $\mathrm{X}(a,b,\theta) = r_{-\theta} \cdot \mathrm{X}(a,b)$. Comme les propriétés de déviations des sommes de Birkhoff sont invariantes par le flot de Teichmüller, nous nous intéressons aux mesures K et A-invariantes sur les composantes de strates. Comme K et A engendrent $\mathrm{SL}(2,\mathbb{R})$, elles sont $\mathrm{SL}(2,\mathbb{R})$-invariantes. De plus, le théorème de Howe-Moore assure :

> T*oute mesure de probabilité* $\mathrm{SL}(2,\mathbb{R})$*-invariante, ergodique sur une composante connexe de strate est également A-ergodique.*

Ainsi, pour une mesure $\mathrm{SL}(2,\mathbb{R})$-ergodique, il est possible d'utiliser le théorème d'Oseledets et appliquer les résultats sur les déviations des sommes de Birkhoff.

4.2 Adhérence des $\mathrm{SL}(2,\mathbb{R})$-orbites des surfaces $\mathrm{X}(a,b)$

Les propriétés génériques des flots linéaires d'une surface de translation peuvent s'étudier au moyen de l'action de $\mathrm{SL}(2,\mathbb{R})$ sur les composantes de strates. Pour utiliser efficacement les théorèmes de théorie ergodique pour cette action, il faut commencer par classifier les mesures $\mathrm{SL}(2,\mathbb{R})$-ergodiques. Dans cette section, nous expliquons cette classification pour les adhérences des orbites des surfaces $\mathrm{X}(a,b)$.

Dans la section 1.4, nous avons vu que la surface $\mathrm{X}(a,b)$ était un revêtement de degré 4 de la surface $\mathrm{L}(a,b)$ (voir figure 1.8 page 7). Le groupe de revêtement $\mathrm{Deck}(\mathrm{X}(a,b)/\mathrm{L}(a,b))$ est isomorphe à $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. On note h (resp. v), l'élément de $\mathrm{Deck}(\mathrm{X}(a,b)/\mathrm{L}(a,b))$ qui échange les copies 00 avec 10 et 01 avec 11 (resp. les copies 00 avec 01 et 10 avec 11). L'action de $\mathrm{SL}(2,\mathbb{R})$ commute avec le passage au quotient $\mathrm{X}(a,b,\theta) \to \mathrm{L}(a,b,\theta)$ et donc, les $\mathrm{SL}(2,\mathbb{R})$-orbites des surfaces $\mathrm{X}(a,b)$ dans $\mathcal{H}(2,2,2,2)$ sont en correspondance avec les $\mathrm{SL}(2,\mathbb{R})$-orbites des surfaces $\mathrm{L}(a,b)$ dans $\mathcal{H}(2)$. Nous avons déjà vu une mesure ergodique sur $\mathcal{H}(2)$: la mesure de Lebesgue (voir la note de bas de page 2 page 17), mais il existe d'autres mesures. Une *surface de Veech* [Vee89] est une surface de translation S telle que son stabilisateur pour l'action de $\mathrm{SL}(2,\mathbb{R})$ est un réseau. Plus simplement, c'est une surface qui admet beaucoup de symétries[1]. Le point important pour notre étude est le fait que la $\mathrm{SL}(2,\mathbb{R})$-orbite d'une surface de Veech est fermée dans sa strate et supporte une unique mesure de probabilité $\mathrm{SL}(2,\mathbb{R})$-invariante.

La condition « être une surface de Veech » est très restrictive. Dans toute strate de genre $g \geq 2$, il y a une quantité non dénombrable de $\mathrm{SL}(2,\mathbb{R})$-orbites et seulement une quantité dénombrable d'entre

[1]Les surface de Veech ont été introduites pour leur propriété de dynamique optimale : une surface de translation S a une *dynamique optimale* si pour toute direction $\theta \in S^1$, le flot linéaire dans la direction θ est soit uniquement ergodique, soit complètement périodique.

elles sont des $\mathrm{SL}(2,\mathbb{R})$-orbites d'une surface de Veech. K. Calta [Cal04] et C. McMullen [McM03, McM05] ont classifié les $\mathrm{SL}(2,\mathbb{R})$-orbites des surfaces de Veech dans la strate $\mathcal{H}(2)$:

> L*a surface* $\mathrm{L}(a,b)$ *est une surface de Veech si et seulement si elle vérifie une des deux conditions suivantes :*
>
> 1. *a et b sont rationnels, auquel cas* $\mathrm{L}(a,b)$ *est revêtement d'un tore ;*
>
> 2. *il existe* x *et* y *deux nombres rationnels et* D *un entier positif sans facteur carré tel que :*
>
> $$\frac{1}{1-a} = x + y\sqrt{D} \qquad \text{et} \qquad \frac{1}{1-b} = (1-x) + y\sqrt{D}.$$
>
> *De plus, chaque* $\mathrm{SL}(2,\mathbb{R})$*-orbite d'une surface de Veech de* $\mathcal{H}(2)$ *contient une surface de la forme* $\mathrm{L}(a,b)$.

C. McMullen [McM07] démontre que s'arrête là la liste des mesures invariantes. Plus précisément :

> L*es seuls fermés* $\mathrm{SL}(2,\mathbb{R})$*-invariants irréductibles de la strate* $\mathcal{H}(2)$ *sont la strate elle-même et les* $\mathrm{SL}(2,\mathbb{R})$*-orbite des surfaces de Veech.*
> *Les seules mesures de probabilité* $\mathrm{SL}(2,\mathbb{R})$*-invariante ergodique de la strate* $\mathcal{H}(2)$ *sont la mesure de Lebesgue sur la strate et les mesures naturelles supportées sur les* $\mathrm{SL}(2,\mathbb{R})$*-orbites des surfaces de Veech.*

Une classification des fermés et mesures invariants existe également dans l'autre strate de genre 2 : $\mathcal{H}(1,1)$; mais, en genre $g \geq 3$, peu de choses sont connues.

Comme l'application $\mathrm{X}(a,b) \mapsto \mathrm{L}(a,b)$ s'étend en une application d'image $\mathcal{H}(2)$, le résultat ci-dessus donne une dichotomie précise pour les adhérences des orbites des surfaces $\mathrm{X}(a,b)$:

> S*oit* a *et* b *deux paramètres entre* 0 *et* 1*. Si* $\mathrm{X}(a,b)$ *est une surface de Veech alors sa* $\mathrm{SL}(2,\mathbb{R})$*-orbite est fermée dans la strate* $\mathcal{H}(2,2,2,2)$ *sinon, sa* $\mathrm{SL}(2,\mathbb{R})$ *orbite est isomorphe à* $\mathcal{H}(2)$*. Dans les deux cas, l'adhérence supporte une unique mesure* $\mathrm{SL}(2,\mathbb{R})$*-invariante.*

4.3 Sommes des exposants de Lyapunov

Nous avons vu dans la section 3.2 que les exposants de Lyapunov du cocycle de Kontsevich-Zorich contrôlaient les déviations des sommes de Birkhoff. En particulier, un exposant de Lyapunov est responsable de la diffusion du vent dans les arbres. Dans cette section, suivant le travail en cours d'A. Eskin, M. Kontsevich et A. Zorich [EKZ], nous évoquons comment le calcul de ces exposants est possible. Notre objectif est de formuler un théorème qui explique la valeur $2/3$ obtenue pour le taux de diffusion du vent dans les arbres.

Soit $T : X \to X$ un système dynamique et μ une mesure ergodique. Rappelons que la limite des moyennes de Birkhoff d'une fonction $f : X \to \mathbb{R}$ s'exprime en terme d'intégrale de la fonction f par rapport à la mesure μ. Dans le cas des exposants de Lyapunov d'un cocycle $A : X \to \mathrm{GL}(d,\mathbb{R})$, il existe une formule similaire : la *formule de Furstenberg* ([BL85] théorème 3.6 ou [Fur02]). Mais cette dernière fait intervenir une mesure stationnaire sur l'espace projectif qui n'est pas directement accessible à partir des données T, μ et A. Il existe cependant des algorithmes d'approximation numérique généralisant les méthodes d'approximation de valeurs propres d'une matrice. Dans les années 1990, M. Kontsevich et A. Zorich, à l'aide de programmes informatiques simulant l'induction de Rauzy, ont constaté des phénomènes de rationalité pour les sommes des exposants de Lyapunov positifs du cocycle de Kontsevich-Zorich. Une explication de ces phénomènes fut donnée par M. Kontsevich [Kon97] et développée dans les travaux de G. Forni [For02, For11] et d'A. Eskin, M. Kontsevich et A. Zorich [EKZ]. Ces auteurs démontrent que, pour la plupart des mesures $\mathrm{SL}(2,\mathbb{R})$-ergodiques sur une strate de surfaces de translation, il existe une formule pour la somme $\nu_1 + \ldots + \nu_g$ des exposants de Lyapunov positifs du cocycle de Kontsevich-Zorich. Cette formule repose sur beaucoup de travaux antérieurs : la classification des composantes connexes de strates ([KZ03], voir la section 2.2), le calcul des volumes de ces composantes ([EO01] et [EOP08]), la description du « bord principal » des composantes des strates et les valeurs des constantes de Siegel-Veech ([EMZ03] et [MZ08]), etc.

Les surfaces $\mathrm{X}(a,b)$ ont la particularité d'être hyperelliptiques : elle possède une symétrie d'ordre 2 qui agit comme $-Id$ sur la structure de translation et dont le quotient est une sphère. Cette symétrie s'appelle la *symétrie hyperelliptique*. Cette propriété est invariante par $\mathrm{SL}(2,\mathbb{R})$ et fermée. Ainsi, les adhérences des $\mathrm{SL}(2,\mathbb{R})$-orbites des surfaces $\mathrm{X}(a,b)$ ne contiennent que des surfaces hyperelliptiques. Pour de tels ensembles, les sommes des exposants ne dépendent pas de la mesure $\mathrm{SL}(2,\mathbb{R})$-ergodique considérée[2]. Ce phénomène, conjecturé dans [Kon97], a été prouvé pour le genre 2 par M. Bainbridge [Bai07, Bai10] et dans le cas général dans [EKZ] :

S*oit* μ *une mesure* $\mathrm{SL}(2,\mathbb{R})$*-ergodique sur une strate* $\mathcal{H}_g(\kappa)$ *provenant du revêtement d'orientation d'une mesure régulière* $\overline{\mu}$ *sur une strate de différentielles quadratiques*[3] *sur la sphère* $\mathcal{Q}(d_1, d_2, \ldots, d_n)$*. Alors, la somme des exposants de Lyapunov positifs* $\nu_1 \geq \ldots \geq \nu_g$ *pour la mesure* μ *est donnée par :*

$$\nu_1 + \ldots + \nu_g = \frac{1}{4} \sum_{\substack{j \text{ tel que} \\ d_j \text{ impair}}} \frac{1}{d_j + 2}.$$

En particulier elle ne dépend que des degrés des zéros et des pôles de la strate quadratique $\mathcal{Q}(d_1, d_2, \ldots, d_n)$ *et pas de la mesure* $\overline{\mu}$*.*

Dans la section 4.2, nous avons vu que les surfaces $\mathrm{X}(a,b)$ construites pour décrire la dynamique du vent dans les arbres ont une symétrie de groupe $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ dont le quotient est $\mathrm{L}(a,b)$. Chaque quotient intermédiaire entre $\mathrm{L}(a,b)$ et $\mathrm{X}(a,b)$ est une surface hyperelliptique pour lequel la somme des exposants de Lyapunov positifs est donnée par le théorème ci-dessus. Il est possible de calculer les valeurs des exposants individuels de $\mathrm{X}(a,b)$ à partir de ces informations sur les sommes partielles. Plus précisément, dans l'annexe B, nous démontrons :

[2] Signalons qu'au-delà des lieux hyperelliptiques, D. Chen et M. Möller [CM] ont démontré la constance des sommes des exposants de Lyapunov pour les mesures supportées sur les $\mathrm{SL}(2,\mathbb{R})$-orbites de surfaces de Veech dans certaines strates de genre $g \leq 5$.

[3] Une différentielle quadratique est une généralisation de surface de translation pour laquelle on autorise les inversions dans le recollement des côtés des polygones. Dans ce cas, les singularités coniques sont des multiples de π (et non plus 2π). La notation $\mathcal{Q}(d_1, d_2, \ldots, d_n)$ dénote la strate de différentielles quadratiques constituée des surfaces dont les singularités coniques ont pour angles $(d_1+1)\pi, (d_2+1)\pi, \ldots, (d_n+1)\pi$. Le genre g de la surface est alors donné par la formule : $4g - 4 = d_1 + d_2 + \ldots d_n$ (comparer avec la section 2.2).

S*oit* μ *une mesure* $\mathrm{SL}(2,\mathbb{R})$*-ergodique sur* $\mathcal{H}_5(2,2,2,2)$ *provenant d'une mesure sur* $\mathcal{H}(2)$ *via le revêtement (topologique)* $\mathrm{X}(a,b) \to \mathrm{L}(a,b)$. *Alors, le cocycle de Kontsevich-Zorich admet quatre sous-espaces invariants donnant la décomposition :*

$$H^1(S;\mathbb{R}) = V^{++} \oplus V^{+-} \oplus V^{-+} \oplus V^{--}.$$

Dans cette décomposition, V^{+-} *désigne l'ensemble des vecteurs h-invariants et v-anti-invariants et les autres composantes sont définies de la même façon. La dimension de* V^{++} *est* 4 *et celle des trois autres sous-espaces est* 2. *Le drapeau d'Oseledets du cocycle de Kontsevich-Zorich préserve cette décomposition et les exposants de Lyapunov positifs associés sont* $1, 1/3$ *pour* V^{++}, $2/3$ *pour* V^{+-} *et* V^{-+} *et* $1/3$ *pour* V^{--}.

Le cocycle $f \in H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ définissant la dynamique du vent dans les arbres est la somme d'une composante horizontale f_h et d'une composante verticale f_v. La composante horizontale est dans l'espace V^{-+} et la verticale dans l'espace V^{+-}. Ainsi, c'est bien l'exposant $2/3$ qui contrôle les sommes de Birkhoff.

4.4 Transversalité des surfaces $\mathrm{X}(a,b)$ au flot de Teichmüller

Il est possible de calculer les valeurs des exposants du cocycle de Kontsevich-Zorich pour toute mesure $\mathrm{SL}(2,\mathbb{R})$-invariante supportée sur l'adhérence d'une orbite d'une surface $\mathrm{X}(a,b)$. En effet, il se diagonalise par blocs et le cocycle f contrôlant la diffusion du vent dans les arbres est contenu dans un bloc de taille 2×2 dont l'exposant positif correspondant est $2/3$. Pour conclure que les sommes de Birkhoff sont contrôlées par cet exposant, il reste à montrer que les surfaces $\mathrm{X}(a,b)$ sont Oseledets génériques.

Ce problème de généricité intervient fréquemment dans l'étude des billards : lorsqu'on utilise un théorème de théorie ergodique, on obtient un résultat générique et rien ne nous permet d'assurer qu'une orbite particulière est effectivement générique. Dans le cas du vent dans les arbres, nous utilisons le fait que l'ensemble des surfaces $\mathrm{X}(a,b,\theta)$ vérifie une propriété de transversalité par rapport au flot de Teichmüller.

D'une part, être Oseledets générique, est une propriété invariante par le flot de Teichmüller. D'autre part, le drapeau d'Oseledets du cocycle de Kontsevich-Zorich vérifie la propriété de ne dépendre que des coordonnées horizontales. Du point de vue des suspensions d'échanges d'intervalles (voir la section 2.1), cela se traduit par le fait que ce drapeau ne dépend que de la coordonnée λ et pas de la coordonnée τ des données de suspension $\zeta = \lambda + i\tau$ (voir la section 3.2). Il suffit maintenant de montrer que la « partie horizontale » des surfaces $\mathrm{X}(a,b,\theta)$ contient toutes les longueurs possibles d'échanges d'intervalles. Pour cela, revenons aux formules donnant les longueurs transverses de l'échange d'intervalles obtenues dans la section 1.4. De ces formules, on déduit facilement qu'à toute donnée (a,b,θ) on associe, à multiplication près par un scalaire, un unique quadruplet de longueurs $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Et réciproquement. Autrement dit, à chaque donnée de longueurs d'échange d'intervalles λ, on associe *une* surface $\mathrm{X}(a,b,\theta)$.

Nous énonçons de manière plus précise le théorème principal de l'annexe B dont nous avons retracé les principaux arguments de la démonstration :

L*es paramètres* a *et* b *désignent des réels entre* 0 *et* 1*. On note* $\phi_T^\theta(x)$ *le flot du billard* $V(a,b)$*.*
Si $\mathrm{X}(a,b)$ *est une surface de Veech, alors, pour un angle* θ *générique :*

$$\limsup_{T\to\infty} \frac{\log d(x,\phi_T^\theta(x))}{\log T} = \frac{2}{3}.$$

Pour des paramètres a *et* b *génériques, pour un angle* θ *générique :*

$$\limsup_{T\to\infty} \frac{\log d(x,\phi_T^\theta(x))}{\log T} = \frac{2}{3}.$$

Les deux cas ci-dessus correspondent aux deux familles de mesure sur la strate $\mathcal{H}(2)$ construites dans la section 4.2. Nous appliquons ensuite le théorème de Fubini pour passer d'un résultat générique sur $\mathcal{H}(2)$ au cas particulier des surfaces $\mathrm{X}(a,b)$. Comme la dimension de l'adhérence est plus petite dans le cas des surfaces de Veech (dimension 3) que dans le cas de la mesure de Lebesgue sur la strate (dimension 7), le résultat est plus précis pour les paramètres a et b correspondants.

Conclusion

Le vent dans les arbres est un cas particulier de surface de translation infinie. Nous replaçons nos résultats dans ce contexte et ouvrons notre travail sur quelques questions. Signalons qu'au-delà des billards infinis périodiques, l'introduction des surfaces de translation infinies est motivée par l'étude des billards irrationnels (voir les articles de F. Valdez [Val09, Val]).

Récurrence des surfaces de translation infinies périodiques

Une surface de translation *infinie périodique* est un revêtement infini d'une surface de translation (finie). Soit S une surface de translation compacte et $\Sigma \subset S$ un ensemble discret. Les revêtements de S ramifiés seulement au-dessus de Σ sont en bijection avec les sous-groupes du groupe fondamental $\pi_1(S\backslash\Sigma)$. Dans le cas où le revêtement est normal et abélien de groupe A, il existe un élément $f \in H^1(S\backslash\Sigma; A) \simeq H_1(S, \Sigma; A)$ tel que ce sous-groupe est donné par le noyau de la composition

$$\pi_1(S\backslash\Sigma) \to H_1(S\backslash\Sigma; \mathbb{Z}) \xrightarrow{f} A.$$

On parle alors de A-revêtement de S. Par exemple, le vent dans les arbres est un $\mathbb{Z}^2$-revêtement de la surface $\mathrm{X}(a, b)$ (voir la section 1.3).

Il existe une distance naturelle dans une surface de translation infinie induite par la métrique plate. Cependant, cette métrique présente le défaut d'autoriser « des bonds » au niveau des singularités coniques. L'escalier infini de [HHW] est un exemple de $\mathbb{Z}$-revêtement de diamètre borné ! Une notion plus maniable que la distance plate est une métrique sur le groupe de revêtement. Pour les $\mathbb{Z}^d$-revêtements, nous utilisons la norme euclidienne sur $\mathbb{Z}^d$. Dans certains cas, comme celui du vent dans les arbres, ces deux notions coïncident[1].

Dans le cas $A = \mathbb{Z}$, la propriété de récurrence découle d'un résultat de K. Schmidt [Sch77] et J.-P. Conze [Con09]. P. Hooper et B. Weiss [HW] démontrent :

> S*oit S une surface de translation compacte et $\tilde{S}$ un $\mathbb{Z}$-revêtement de S déterminée par $f \in H_1(S, \Sigma; \mathbb{Z})$. On note ϕ_t^θ (resp. $\tilde{\phi}_t^\theta$) le flot linéaire de $r_{-\theta}S$ (resp. $r_{-\theta}\tilde{S}$). Alors les conditions suivantes sont équivalentes :*
>
> 1. *le revêtement est sans biais (autrement dit, f est de moyenne nulle pour ω) ;*
> 2. *pour presque tout θ le flot linéaire $\tilde{\phi}_t^\theta$ est récurrent ;*
> 3. *pour tout θ tel que ϕ_t^θ est ergodique, $\tilde{\phi}_t^\theta$ est récurrent.*

Comme nous l'avons remarqué dans la section 1.5, dans le cas général des $\mathbb{Z}^d$-revêtements abéliens, la condition de moyenne nulle sur le cocycle est nécessaire mais plus suffisante. Une surface infinie

[1] Plus généralement, ces deux notions coïncident si et seulement si il existe une borne sur l'ordre des ramifications au-dessus des points de Σ. C'est en particulier le cas si le revêtement n'est pas ramifié.

$\tilde{S}$ est dite *récurrente* (resp. *divergente*) si, pour presque toute direction θ, le flot linéaire dans la direction θ de $\tilde{S}$ est récurrent (resp. divergent). N. Chevallier et J.-P. Conze [CC09] montrent que si l'exposant polynomial de diffusion des sommes de Birkhoff est inférieur à $1/d$, le flot est nécessairement récurrent. Ainsi, en utilisant notre résultat sur les déviations contenu dans l'annexe B nous obtenons :

> S*oit S une surface de Veech et Σ un sous-ensemble fini de l'ensemble de ses points périodiques. Soit $f \in H_1(S, \Sigma; \mathbb{Z}^d)$ un cycle relatif et $\tilde{S}$ le $\mathbb{Z}^d$-revêtement infini associé. On suppose que l'action du cocycle de Kontsevich-Zorich sur $H^1(S; \mathbb{R})$ préserve un sous-espace E dans presque toute direction θ et que $f \in E$. Si les exposants de Lyapunov du cocycle de Kontsevich-Zorich restreint à E sont tous strictement inférieurs à $1/d$ alors $\tilde{S}$ est une surface infinie récurrente.*

En utilisant des revêtements cycliques de la sphère ([EKZ11] et [FMZ11]) il est possible de donner des exemples de surfaces $\mathbb{Z}^d$-périodiques récurrentes avec d arbitrairement grand. Dans la suite de cette section, on entend par revêtement infini d'une surface de Veech un revêtement ramifié seulement au-dessus de points périodiques comme dans l'énoncé ci-dessus.

Pour le vent dans les arbres, malgré un exposant polynomial de diffusion égal à $2/3$, la surface infinie est récurrente pour certains paramètres a et b (voir section 1.5). Vu la spécificité de la preuve dans ce cadre, nous nous demandons dans quelle mesure ce résultat s'étend à d'autres revêtements $\mathbb{Z}^2$-périodique. En particulier :

> **Question:** *Etant donné $\nu < 1$, peut-on trouver un $\mathbb{Z}^2$-revêtement sans biais $\tilde{S}$ d'une surface de Veech vérifiant les deux conditions suivantes :*
>
> - *le taux polynomial de diffusion dans $\tilde{S}$ est supérieur à ν ;*
> - *$\tilde{S}$ est récurrent ?*

Bien que, pour une direction générique, le flot linéaire du vent dans les arbres soit récurrent, nous avons démontré l'existence de directions bien spécifiques pour lesquelles le flot du billard est divergent. Cependant, existe t-il des situations pour lesquelles la récurrence n'est pas générique ?

> **Question:** *Existe t-il un $\mathbb{Z}^2$-revêtement sans biais d'une surface de Veech qui soit non récurrent ? divergent ?*

Problème de généricité pour le taux de diffusion

Nous aurions aimé supprimer l'hypothèse de généricité des paramètres a et b de notre théorème sur le taux de diffusion du vent dans les arbres. Cette généricité est imposée par l'utilisation du théorème d'Oseledets pour le cocycle de Kontsevich-Zorich. J. Athreya et G. Forni [AF08], en utilisant d'une part la géométrie du flot de Teichmüller et d'autre part des estimations sur la norme de Hodge[2], démontrent que les déviations de moyennes ergodiques sont toujours polynomiales :

[2] La *norme de Hodge* permet d'étudier le cocycle de Kontsevich-Zorich de manière plus algébrique. L'idée d'introduire la norme de Hodge est due à M. Kontsevich [Kon97] et est largement développée dans l'article de G. Forni [For02].

S*oit $g > 1$ un entier. Il existe une constante $\alpha < 1$ telle que pour toute surface de translation S de genre g, pour tout cocycle f de $H^1(S;\mathbb{Z})$, pour un angle θ générique, le taux de diffusion de f pour le flot linéaire dans la direction θ sur S est au plus α. Plus précisément, les sommes de Birkhoff $S_T(f,x)$ de f le long de l'orbite de $x \in S$ dans la direction θ vérifient :*

$$\limsup_{T\to\infty} \frac{\log |S_T(f,x) - Tm(f,\theta)|}{\log T} \leq \alpha$$

où $m(f,\theta)$ est la moyenne de f dans la direction θ.

La force de ce théorème est qu'il s'applique à *toute* surface de translation. On peut espérer un résultat d'équidistribution pour le théorème d'Oseledets dans le sens suivant :

Question: *Soit S une surface de translation. Supposons que l'adhérence de sa* $\mathrm{SL}(2,\mathbb{R})$-*orbite supporte une unique mesure μ,* $\mathrm{SL}(2,\mathbb{R})$*-ergodique et de support total. Pour une direction θ générique, le cocycle de Kontsevich-Zorich le long de l'orbite de $r_\theta \cdot S$ est-il Oseledets générique pour la mesure μ ?*

En particulier :

Question: *Pouvons-nous supprimer l'hypothèse de généricité des paramètres a et b dans le théorème sur le taux de diffusion du vent dans les arbres ?*

Champ de vecteur affine et drapeau d'Oseledets

Le troisième problème que nous abordons, concerne une forme de régularité du cocycle de Kontsevich-Zorich. Dans le cadre du vent dans les arbres, ce dernier se diagonalise par blocs, le bloc contrôlant la diffusion du vent dans les arbres étant de taille 2×2. En particulier, aucune ambiguïté n'est possible sur l'exposant de Lyapunov qui contrôle la diffusion. Mais qu'en est-il pour un facteur irréductible du cocycle de Kontsevich-Zorich de taille supérieure ?

Soit S une surface (topologique) de genre g. Nous notons $\Omega\mathcal{T}_S(\kappa)$ une strate de l'espace de Teichmüller et $\Omega\mathcal{M}(\kappa)$ la strate correspondante de l'espace des modules[3]. Soit μ une mesure $\mathrm{SL}(2,\mathbb{R})$-ergodique sur $\Omega\mathcal{M}(\kappa)$. On note $1 = \nu_1 > \nu_2 > \ldots > \nu_k$ les exposants de Lyapunov positifs du cocycle de Kontsevich-Zorich pour la mesure μ. Pour une structure de translation $\omega \in \Omega\mathcal{T}(\kappa)$ Oseledets générique, on note $V_i^u(\omega) \subset H^1(S;\mathbb{R})$ (resp. $V_i^s(\omega) \subset H^1(S;\mathbb{R})$ et $V^c(\omega) \subset H^1(S;\mathbb{R})$) l'espace instable associé à ν_i (resp. l'espace stable associé à $-\nu_i$ et l'espace central). Définissons :

$$U_i(\omega) = V^c(\omega) \bigoplus_{j=0}^{i} (V_j^s(\omega) \oplus V_j^u(\omega)).$$

Pour un champ de vecteurs affine $f : \Omega\mathcal{T}(\kappa) \to H^1(S;\mathbb{R})$ et une structure de translation $\omega \in \Omega\mathcal{T}(\kappa)$ Oseledets générique, posons :

$$\nu(f,\omega) = \min_{i\in\{0,1,\ldots,k\}} \{\nu_i |\ f(\omega) \in U_i(\omega)\}.$$

Autrement dit, $\nu(f,\omega)$ est le coefficient qui contrôle les déviations de f le long de l'orbite de ω.

[3]Cette approche plus algébrique des espaces des surfaces de translation est nécessaire pour utiliser la norme de Hodge. Les strates $\mathcal{H}(\kappa)$ que nous avons vues correspondent à $\Omega\mathcal{M}(\kappa)$ dont chaque point correspond à une classe d'isomorphisme de surface de translation. Une strate de l'espace de Teichmüller $\Omega\mathcal{T}_S(\kappa)$ est un espace « déplié » sur lequel $\mathrm{SL}(2,\mathbb{R})$ agit librement. L'espace $\Omega\mathcal{M}(\kappa)$ est le quotient de $\Omega\mathcal{T}_S(\kappa)$ par un groupe discret : le *groupe modulaire*. Ce dernier est relié de très près à l'induction de Rauzy.

Nous avons, dans la section 3.2, utilisé un champ de vecteurs affine. En effet, la somme de Birkhoff $S_N(f,T,x)$ d'une fonction f au-dessus d'un échange d'intervalles T correspond à un champ de vecteurs constant. La taille de cette somme est de l'ordre de $N\lambda_A$ et en particulier croit à vitesse linéaire : l'exposant de Lyapunov qui contrôle cette somme est $\nu_1 = 1$. La différence $S_N(T,\chi_A,x) - N\lambda(f)$ correspond à un champ de vecteurs affine à cause de la dépendance en λ. Plus généralement, si $f : \Omega\mathcal{T}(\kappa) \to H^1(S;\mathbb{R})$ est un champ de vecteurs constant alors $\omega \mapsto f - \omega(f)$ est le champ de vecteurs affine sur $\Omega\mathcal{T}_S(\kappa)$ qui mesure les déviations des sommes de Birkhoff de f.

> **Question:** *Soit μ une mesure $\mathrm{SL}(2,\mathbb{R})$-ergodique et $f : \Omega\mathcal{T}(\kappa) \to H^1(S;\mathbb{R})$ un champ de vecteurs affine sur une strate de l'espace de Teichmüller. La foncétion $\nu(f,.)$ est-elle μ-presque partout constante ?*

La fonction $\nu(f,.)$ est g_t invariante sur l'espace de Teichmüller mais *n'est pas* invariante pour l'action du groupe modulaire. Ainsi, il n'est pas possible d'utiliser directement l'ergodicité du flot de Teichmüller sur l'espace des modules $\Omega\mathcal{M}(\kappa)$ pour répondre positivement à cette question.

Revêtements non abéliens

La surface associée au vent dans les arbres est un $\mathbb{Z}^2$-revêtement (sans biais) d'une surface de translation compacte. Dans cette partie, nous nous interrogeons sur la récurrence et la diffusion des surfaces de translation infinies périodiques dont le groupe de revêtement n'est plus abélien.

Du point de vue des échanges d'intervalles, un revêtement non-abélien devient une somme de Birkhoff non commutative, autrement dit un *cocycle*. Soit π une permutation sur l'alphabet $\mathcal{A}$ de cardinal n. Soit G un groupe et g_α pour $\alpha \in \mathcal{A}$ des éléments de G qui l'engendrent. À toute donnée de longueurs $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}}$ on associe un cocycle (non commutatif) au-dessus de l'échange d'intervalles $T_{\pi,\lambda}$ en considérant la fonction qui vaut g_α sur l'intervalle I_α. Sur le groupe G, on considère la métrique des mots induite par les générateurs g_α. Le théorème ergodique sous-additif de Kingman (généralisant le théorème de Birkhoff et celui d'Oseledets) implique qu'il existe un *taux de diffusion* (ou *vitesse de fuite*) pour ce cocycle. Dans le cas où G est un groupe abélien, cette vitesse de fuite est un exposant de Lyapunov du cocycle de Kontsevich-Zorich.

> **Question:** *Que peut-on dire du taux de diffusion d'un cocycle non commutatif au-dessus d'un échange d'intervalles $T_{\pi,\lambda}$ générique ? Est-il possible de généraliser la décomposition donnée par le théorème d'Oseledets (« phénomène de Kontsevich-Zorich ») ?*

Même dans un cadre non abélien, il est toujours possible d'utiliser la renormalisation. Dans sa version combinatoire, le cocycle de Kontsevich-Zorich non abélien consiste non plus à regarder la suite de matrices produite par l'induction de Rauzy (ou Ferenczi-Zamboni), mais la suite de substitutions. Au-delà des revêtements infinis non-abéliens, l'introduction de ces substitutions est également motivées par l'étude fine des sommmes de Birkhoff. Par exemple, notre preuve de l'existence de trajectoires divergentes pour le vent dans les arbres les utilise.

En relation avec cette question et les exemples construits plus haut de surfaces infinies récurrentes :

> **Question:** *Existe t-il un G-revêtement d'une surface de translation compacte récurrente avec G infini non virtuellement abélien ?*

Une réponse positive est apportée à cette question dans des travaux en cours de J. Cabrol [Cab]. Il construit un exemple avec le groupe de Heisenberg $H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$:

I*l existe un* H*-revêtement récurrent d'un tore.*

En utilisant les mêmes techniques, il démontre le théorème suivant :

S*oit* G *un groupe d'exposant fini*[4]*, alors tout* G*-revêtement est récurrent.*

En particulier le théorème s'applique aux groupes de Burnside et contraste avec la situation des marches aléatoires dont les incréments sont indépendants. De la même façon, nous nous interrongeons sur une famille de groupes qui satisferait a un énoncé opposé.

Question: *Existe t-il un groupe* G *tel que tout* G*-revêtement d'une surface de translation compacte soit divergent ?*

[4]Un groupe G est d'exposant fini s'il existe n tel que pour tout g dans G, $g^n = 1$.

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Annexe A

Trajectoires divergentes du vent dans les arbres

Dans cette première annexe, nous reproduisons une version de l'article sur la construction de trajectoires divergentes du vent dans les arbres. La numérotation des pages tout au long de l'article suit celle de l'article et non de la thèse. Ainsi, la page 53 de cette thèse est numérotée 1. Cette numérotation spéciale sera en italique.

Divergent trajectories in some periodic wind-tree models

Abstract

The periodic wind-tree model is a family $T(a,b)$ of billiards in the plane in which identical rectangular scatterers of size $a \times b$ are disposed at each integer point. It was proven by P. Hubert, S. Lelièvre and S. Troubetzkoy that for a residual set of parameters (a,b) the billiard flow in $T(a,b)$ is recurrent in almost every direction. We prove that for many parameters (a,b) there exists a set $\Lambda \subset S^1$ of positive Hausdorff dimension such that for every $\theta \in \Lambda$ every billiard trajectory in $T(a,b)$ with initial angle θ is divergent.

Résumé

Trajectoires divergentes pour vent dans les arbres

Le "vent dans les arbres" est une famille de billards infinis $T(a,b)$ définis de la manière suivante. Dans le plan euclidien $\mathbb{R}^2$, on place des rectangles de taille $a \times b$ à chaque point entier. Une particule (identifiée à un point) se déplace en ligne droite et rebondit de manière élastique sur les obstacles. P. Hubert, S. Lelièvre et S. Troubetzkoy ont démontré qu'il existait un G_δ dense de paramètres (a,b) pour lesquels, dans presque toute direction $\theta \in S^1$, le flot du billard $T(a,b)$ dans la direction θ est récurrent. Nous prouvons que pour certains paramètres (a,b), il existe un ensemble $\Lambda \subset S^1$ de mesure de Hausdorff positive tel que pour tout $\theta \in \Lambda$ toute trajectoire dans le billard $T(a,b)$ dont l'angle de départ est θ est divergente.

1 Introduction

We study periodic versions of the wind-tree model introduced by P. Ehrenfest and T. Ehrenfest in 1912 [EhEh90]. A point ("wind") moves in the plane and collides with rectangular scatterers ("trees") with the usual law of reflexion. In the periodic version of the wind-tree model, due to J. Hardy and J. Weber [HaWe80], the scatterers are identical rectangular obstacles located periodically in the plane, every obstacle centered at each point of $\mathbb{Z}^2$. The scatterers are rectangles of size $a \times b$, with $0 < a < 1$, $0 < b < 1$. We denote by $T(a, b)$ the subset of the plane obtained by removing the obstacles and name its billiard *the wind-tree model*. Our aim is to understand some of its dynamical properties (see Figure 1 for two different behaviors in the golden wind-tree table $T(\Phi, \Phi)$).

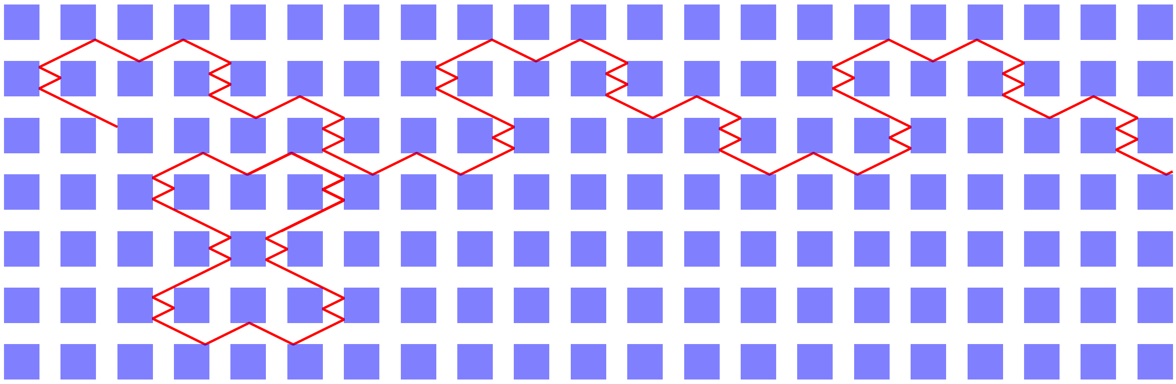


Figure 1: A periodic and a divergent orbits in a half-divergent direction of the "golden wind-tree model" with parameters $a = b = \Phi = \frac{\sqrt{5}-1}{2}$.

The phase space of the billiard is naturally $T(a, b) \times S^1$. Each barrier in $T(a, b)$ is either horizontal or vertical. Hence, for the point $(x, \theta) \in T(a, b) \times S^1$ and for every time t, the possible angles for the orbit of (x, θ) in $T(a, b)$ at time t are θ, $-\theta$, $\pi/2 - \theta$ or $-\pi/2 + \theta$. Let $\tau_h : \theta \mapsto -\theta$ and $\tau_v : \theta \mapsto \pi/2 - \theta$ be respectively the horizontal and vertical reflexions and $K = \langle \tau_h, \tau_v \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ the group they generate. We define the *billiard flow in direction* θ in $T(a, b)$ to be the map $\phi_t^\theta : T(a, b) \times K \to T(a, b) \times K$ which is defined as follows. Let (x_0, κ_0) be an element of $T(a, b) \times K$, then $(x_t, \kappa_t) = \phi_t^\theta(x, \kappa)$ is such that if a ball has an initial position x_0 and an initial angle $\kappa_0(\theta)$ then after time t it has position x_t and direction $\kappa_t(\theta)$. We will often consider the quantity $\phi_t^\theta(x, \tau)$ as an element of $T(a, b)$ and write $\phi_t^\theta(x)$ for $\phi_t^\theta(x, id)$.

Let d be the Euclidean distance in $\mathbb{R}^2$. We say that the flow in direction θ is *recurrent*, if for almost all points x in $T(a, b)$ we have $\liminf_{t\to\infty} d(x, \phi_t^\theta(x)) = 0$. We say that it is *divergent* if for almost all points $x \in T(a, b)$ $\liminf_{t\to\infty} d(x, \phi_t^\theta(x)) = +\infty$. P. Hubert, S. Lelièvre and S. Troubetkoy [HuLeTr] exhibit a residual set $\mathcal{E} \subset (0, 1) \times (0, 1)$ such that for any parameters $(a, b) \in \mathcal{E}$ for almost all $\theta \in S^1$ (with respect to the Lebesgue measure on S^1) the flow in $T(a, b)$ in direction θ is recurrent. In the present paper we study the opposite behavior: the set of parameters (a, b, θ) for which the flow in $T(a, b)$ in direction θ is divergent. As a consequence of our main result (Theorem 2) we obtain

Theorem 1. *If a and b are either rational or quadratic of the form $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x)+y\sqrt{D}$ with x and y rationals and $D > 1$ a square-free integer, then there exists a dense set $\Lambda \subset S^1$ of Hausdorff dimension not smaller than $1/2$ such that for every $\theta \in \Lambda$ and every point x in $T(a, b)$ with infinite forward orbit* $\liminf d(x, \phi_t^\theta(x)) = \infty$. *In particular the flow ϕ_t^θ is divergent.*

The subset Λ which appears in the above theorem is made explicit in Proposition 6.

The strategy to prove Theorem 1 is very similar to the one used in [HuLeTr]. We explain the idea for the special case $T(1/2, 1/2)$. For every angle $\theta \in (0, \pi/2)$ for which the slope $\tan\theta$ is rational, the billiard flow in $T(a, b)$ in direction θ has a periodic behavior. Two important types of slopes are of interest for our purpose: *half-divergent slope* and *periodic slope* (see Figure 1 for an example of a half-divergent slope). We explain the definition on two examples. In the horizontal direction $\theta = 0$ in $T(1/2, 1/2)$, there is a bunch of trajectories which reflect between two consecutive scatterers spaced by $(1, 0)$ while the others go to infinity: 0 is a half-divergent slope. On the contrary, in the direction $\theta = \pi/4$, all trajectories are periodic with the same period: the slope $\tan(\pi/4) = 1$ is periodic. To prove recurrence, the strategy of [HuLeTr] consists in approximate a generic slopes by rational ones which correspond to directions of periodic type in $T(1/2, 1/2)$. To build divergent trajectories in the same billiard table, we use slopes are in a sense badly approximate by slopes of periodic type.

The proof of our main result uses a renormalization algorithm due to S. Ferenczi and L. Zamboni [FeZa10, FeZa]. Their induction operate on interval exchange transformations and we give a geometric interpretation on translation surfaces using suspensions. Similar geometric renormalization is described by C. Ulcigraï and J. Smillie for the regular "octagon" [SmUl11]. The geometric interpretation we use was known in greater generality by P. Hubert and C. Ulcigraï [HuUl].

We first consider a discretization of the flow ϕ_t^θ in $T(a, b)$ and prove that the distance $d(x, \phi_t^\theta(x))$ corresponds to a Birkhoff sum of a function over an interval exchange transformation $g = g_{a,b,\theta}$. Then we build a set of parameters (a, b, θ) by imposing some conditions in the Ferenczi-Zamboni induction of g. For those parameters, we have a very simple continued fraction algorithm-like which is define as follows. For a 4-tuple of positive real numbers $Z = (x_1, x_2, y_1, y_2)$ define

$$F(Z) = \left(y_1,\ y_2,\ x_1 - \left\lfloor \frac{x_1}{y_1 + y_2} \right\rfloor (y_1 + y_2),\ x_2 - \left\lfloor \frac{x_2}{y_2} \right\rfloor y_2 \right)$$

and

$$m(Z) = \left\lfloor \frac{x_1}{y_1 + y_2} \right\rfloor \quad \text{and} \quad n(Z) = \left\lfloor \frac{x_2}{y_2} \right\rfloor$$

where $\lfloor . \rfloor$ designs the floor. If Z satisfies

$$x_1 + x_2 > y_1 > x_2 \quad \text{and} \quad y_1 + y_2 > x_1 > y_2 \tag{1}$$

then $F^2(Z) = Z$. We say that the quadruple Z is *F-renormalizable* if for all $k \geq 0$, $F^k(Z)$ does not satisfy (1). To a F-renormalizable quadruple Z we associate an infinite sequence of 2-tuples $((m_k, n_k))_{k\in\mathbb{N}}$ defined by $m_k(Z) = m(F^k(Z))$ and $n_k(Z) = n(F^k(Z))$. We call the sequence $((m_k(Z), n_k(Z))_{k\geq 0}$ the *F-convergents* of Z. The set of F-renormalizable quadruples defines an uncountable set of zero Lebesgue measure.

Proposition 1. *If $Z = (x_1, x_2, y_1, y_2)$ is F-renormalizable then*

– for $k \geq 0$, if $m_k = 0$ then $m_{k+1} \neq 0$ and $n_{k+1} = 0$,

– for infinitely many k, $m_{2k} \neq 0$. The same is true for m_{2k+1}, n_{2k} and n_{2k+1}.
Conversely, if $((m_k, n_k))_{k\geq 0}$ is a sequence of 2-tuples of non negative integers that satisfy the above condition, then there exists at least one quadruple Z such that for all k, $m_k(Z) = m_k$ and $n_k(Z) = n_k$.

Using the above description of F-renormalizable slopes, our main result is

Theorem 2. *Let $Z = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4_+$ and $(a, b, \theta) \in (0, 1)^2 \times [0, \pi/2)$ be related by*

$$x_1 = (1 - b)\cos\theta \quad x_2 = b\cos\theta \quad y_1 = (1 - a)\sin\theta \quad \text{and} \quad y_2 = a\sin\theta.$$

Assume that Z is F-renormalizable and let $((m_k, n_k))_{k\geq 0}$ be its F-convergents. If for all k, $n_k \equiv 0 \mod 2$, then any infinite forward trajectory in direction θ in $T(a, b)$ is self-avoiding. In particular the flow in direction θ in $T(a, b)$ is divergent.

The infinite billiard $T(a, b)$ can be considered as a particular case of $\mathbb{Z}^2$-periodic translation surface (with finite quotient). For $\mathbb{Z}$-periodic translation surfaces the recurrence of the flow follows from general results on 1-dimensional cocycles. For highly symmetric examples, the ergodicity of $\mathbb{Z}$-periodic translation surface the flow can be established (see [HuSc10], [HuWe], [HoHuWe]). The main difficulty of the wind-tree model comes from dimension 2.

We mention other results on the wind-tree-model. As we explained above, it is proven in [HuLeTr] that for a residual set of parameters (a, b) for almost all angles θ the flow ϕ^θ_t in $T(a, b)$ is recurrent. The problem of diffusion is studied in [DeHuLe] (it is proven that for almost all parameters and almost all angles θ the polynomial growth of $d(p, \phi^\theta_t(p))$ is 2/3). The ergodic decomposition for irrational parameters (a, b) and some angles θ with rational slopes is done in [CoGu]. We would also mention that it follows from the main result in [Ho], the classification of periodic directions in [HuLeTr] and Proposition 7 that

Theorem 3. *If $a = p/q$ and $b = r/s$ are rationals with p, r odd and r, s even, then there exists a dense set $\Lambda \subset S^1$ of Hausdorff dimension not less than $1/2$ such that for every $\theta \in \Lambda$ the billiard flow ϕ^θ_t is ergodic.*

The paper is organized as follows. In Section 2 we define translation surfaces and interval exchange transformations. We build the discretization of the flow ϕ^θ_t as a $\mathbb{Z}^2$-cocycle over an interval exchange transformation. Next in Section 3 we recall the Ferenczi-Zamboni induction and see the relation with the map F defined above. The proof that Theorem 2 implies Theorem 1 is left to Section 3.4. The proof uses the classification of Veech surfaces in genus 2 by K. Calta [Ca04] and C. McMullen [Mc03]. The proof of Theorem 2 is postponed to Section 3.5.

Aknowledgments: The author would like to thank Pascal Hubert and Samuel Lelièvre for introducing him to the known results about the wind-tree model $T(p/q, r/s)$ and more generally to the theory of finite and infinite square tiled surfaces. Many experimentations have been done with the math software Sage [Sa, SaC]. The script (for computations and drawings) as well as a collection of pictures are available on the web page of the author.

2 The wind-tree cocycle

In this section we build a discretization of the billiard flow in $T(a,b)$.

2.1 Translation surface and Poincare maps of the linear flow

A *flat surface* is a compact oriented surface X endowed with a flat metric defined on $X \backslash \Sigma$ where $\Sigma \subset X$ is a finite set of points which are conic type singularities for the metric. It is a *translation surface* if moreover the holonomy given by parallel transport in $X \backslash \Sigma$ is trivial. Concretely, any translation surface can be built from a finite set of polygons P_i in $\mathbb{R}^2$ and identifying pairs of edges with translations. We refer to the survey of A. Zorich [Zo06] and the notes of M. Viana [Vi] for the latter construction and other equivalent definitions of translation surfaces.

Let X be a translation surface, the absence of holonomy implies that directions are globally defined. The geodesic flow on the tangent bundle of X preserves directions and can be defined on X as soon as we specify the direction and the speed. We assume that in a translation surface a fixed direction is given which we call *vertical*. The *linear flow* of X is the unit speed geodesic flow in the vertical direction on X. The flow in a rational billiards can be transformed into the linear flow of a translation surface. We will use this construction to define the wind-tree cocycle (see Section 2.5).

In this paper we mainly focus on the family of translation surfaces $\mathrm{L}(a,b)$ where $0 < a < 1$ and $0 < b < 1$ are two parameters. The surface $\mathrm{L}(a,b)$ is built as follows (see also Figure 2). Take the polygon with vertices $(0,0)$, $(1,0)$, $(1,1-b)$, $(1-a,1-b)$, $(1-a,1)$, and $(0,1)$ and identify the following pairs of edges

- $[(0,0),(1-a,0)]$ with $[(0,1),(1-a,1)]$ (labelled h_1 in Figure 2),
- $[(0,0),(0,1-b)]$ with $[(1,0),(1,1-b)]$ (labelled v_1),
- $[(1-a,0),(1,0)]$ with $[(1-a,1-b),(1,1-b)]$ (labelled h_2) and
- $[(0,1-b),(0,1)]$ with $[(1-a,1-b),(1-a,1)]$ (labelled v_2).

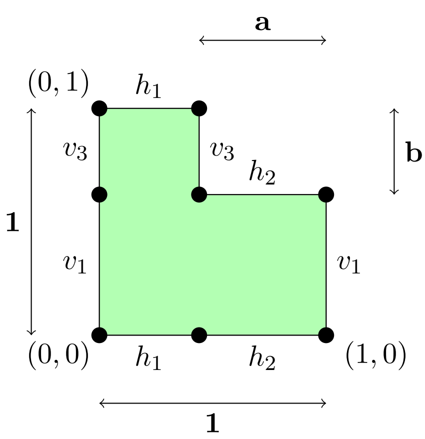


Figure 2: The surface $\mathrm{L}(a,b)$ (letters h_i and v_i indicate gluings).

The translation surface $\mathrm{L}(a,b)$ is a genus 2 surface that has one conic singularity of angle 6π. The equivalence classes of translation surfaces with a single conic singularity of angle 6π form a moduli space denoted $\mathcal{H}(2)$. The number 2 in $\mathcal{H}(2)$ does not refer to the genus but to the *degree* of the singularity. Any surface in $\mathcal{H}(2)$ can be built from a polygon which is L-shaped but non necessarily right-angled (see [Ca04], [HuLe06] and [Mc03]).

Let X be a translation surface and $I \subset X$ an horizontal segment (or any segment transverse to the linear flow of X). The first return map on I is defined for every point in I for which the orbit under the linear flow returns to I before reaching a singularity of the metric. There is a natural Lebesgue measure on I induced from the flat metric of

X which is preserved by the first return map. Out of the discontinuities, the first return map on I is a translation. For our purpose, it will be easier to work with Poincare maps obtained on transversal made of more than one interval.

2.2 Interval exchange transformations and quadrangulations

We mainly follows the presentation of [FeZa, FeZa10] and give a geometric point of view on their construction. We use here the letters ℓ (for left) and r (for right) whereas in [FeZa, FeZa10] the letters m (for minus) and p (for plus) are used.

Let $\lambda = ((\lambda_{1,\ell}, \lambda_{1,r}), \ldots, (\lambda_{d,\ell}, \lambda_{d,r}))$ be a vector of d pairs of real numbers where $\lambda_{i,\ell} < 0$ and $\lambda_{i,r} > 0$. Set $E_i =]\lambda_{i,\ell}, \lambda_{i,r}[$ for $i = 1, \ldots, d$. We define a map T on the disjoint union $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_d$ for which the origin of every interval E_i is a discontinuity of T^{-1}. The combinatorics of T depends on a pair of permutations $\pi = (\pi_\ell, \pi_r) \in S_d \times S_d$ such that the group they generate acts transitively on $\{1, \ldots, d\}$. We build two decompositions of each interval E_i as follows. Let $E_{i,\ell} =]\lambda_{i,\ell}, 0[$ and $E_{i,r} =]0, \lambda_{i,r}[$ (which corresponds to the past) and $E_{i,L} =]\lambda_{i,\ell}, \lambda_{i,\ell} + \lambda_{\pi_\ell(i),r}[$, $E_{i,R} =]\lambda_{i,r} + \lambda_{\pi_r(i),\ell}, \lambda_{i,r}[$ (which corresponds to the future). The map $T : E \to E$ is such that each restriction of T to $E_{i,L}$ (resp. $E_{i,R}$) is a translation onto $E_{\pi_\ell(i),r}$ (resp. $E_{\pi_r(i),\ell}$). We assume implicitly here, that the length-vector λ satisfies the *train-track relations*

$$\lambda_{i,r} - \lambda_{i,\ell} = \lambda_{\pi_\ell(i),r} - \lambda_{\pi_r(i),\ell} \qquad \text{for } i = 1, \ldots, d. \tag{2}$$

We denote by $T_{\pi,\lambda}$ the application constructed above from the data λ and $\pi = (\pi_\ell, \pi_r)$ (see the top picture of Figure 4). We call the map $T : E \to E$ an *interval exchange transformation.* We warn the reader that our definition of interval exchange transformation does not correspond to the standard one in which one interval is cut in several pieces. In our case many intervals are cut in only two pieces.

Let $T_{\pi,\lambda} : E \to E$ be an interval exchange transformation. There is a natural way to code the orbits of the dynamical system associated to T. To each point x in E we associate a label in $\mathcal{A} = \{1, \ldots, d\} \times \{\ell, r\}$ corresponding to the interval $E_{i,\ell}$ or $E_{i,r}$ it belongs. To an infinite orbit x, $T(x)$, $T^2(x)$, ... we associate an infinite sequence $w \in \mathcal{A}^{\mathbb{N}}$ in such way that w_k is the label associated to the point $T^k(x)$. The orbit of x starts with the finite word $w = w_0 w_1 w_2 \ldots w_{N-1} \in \mathcal{A}^N$ if and only if it belongs to the interval

$$E_w = \bigcap_{k=0,1,\ldots,N-1} T^{-k}(E_{w_k}).$$

Similarly to Veech zippered-rectangle construction [Ve82], we define zippered rectangles for the map T.

Definition 1. *A suspension datum for an interval exchange transformation* (λ, π) *is a vector* $\zeta = (\zeta_{i,\ell}, \zeta_{i,r}) \in (\mathbb{C}^2)^d$ *such that*

– $\mathrm{Re}(\zeta_{i,\ell}) = \lambda_{i,\ell}$ *and* $\mathrm{Re}(\zeta_{i,r}) = \lambda_{i,r}$*,*

– $\mathrm{Im}(\zeta_{i,l}) > 0$ *and* $\mathrm{Im}(\zeta_{i,r}) > 0$*,*

– $\zeta_{i,r} - \zeta_{i,\ell} = \zeta_{\pi_\ell(i),r} - \zeta_{\pi_r(i),\ell}$ *(*train-track relations for a suspension*).*

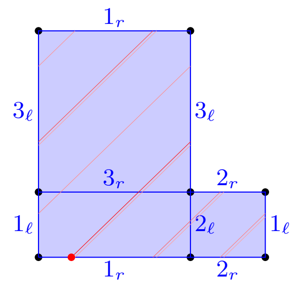
To a suspension datum ζ of (π, λ) we associate a translation surface in the following way. For $i = 1, 2, \ldots, d$, let R_i be the quadrilateral with vertices (in trigonometric order)

$\zeta_{i,l}$, 0, $\zeta_{i,r}$, $\zeta_{i,r}+\zeta_{\pi_r(i),\ell}=\zeta_{i,\ell}+\zeta_{\pi_\ell(i),r}$. The suspension $S(\pi,\zeta)$ is the disjoint union of the rectangles R_i for $i=1,\dots,d$ in which we identify the sides which have the same label (i,ℓ) or (i,r) (see the second picture in Figure 4).

Definition 2. *Let X be a translation surface. A* quadrangulation *of X is a simplicial decomposition of X for which the vertices are the conic singularities of X, the edges are geodesics and every face is a quadrilateral which does not contain any singularity. A quadrangulation is* admissible *if every face has exactly two adjacent edges for which the linear flow (in the vertical direction) is incoming.*

In the suspension $S(\pi,\zeta)$ of an interval exchange transformation, the rectangles R_i naturally define an admissible quadrangulation. Reciprocally any admissible quadrangulation gives rise to a train-track: we associate to a quadrangulation the first return map on its sides (see the third picture of Figure 4).

Let X be a translation surface with an admissible quadrangulation. To an orbit of the linear flow, we associate its natural *cutting sequence* made of the ordered list of edges meet by the orbit. The cutting sequences in a suspension $S(\pi,\zeta)$ of an interval exchange transformation $T_{\pi,\lambda}$ are in bijections with the coding of the orbits in $T_{\pi,\lambda}$.



$1_r\, 3_r\, 3_l\, 1_r\, 2_l\, 2_r\, 1_l\, 3_r\, 3_l\, 1_r\, 3_r\, 3_l\, 1_r\, 2_l\, 2_r \dots$

Figure 3: A geodesic in $\mathrm{L}(a,b)$ and its cutting sequence.

2.3 The quadrangulation of $\mathrm{L}(a,b)$

We are now interested in the cutting sequences of the linear flow in the surfaces $r_{-\theta}\cdot\mathrm{L}(a,b)$ defined in Section 2.1 and where $r_{-\theta}$ denotes the rotation by an angle θ. The linear flow in the surface $r_{-\theta}\cdot\mathrm{L}(a,b)$ is the geodesic flow of $\mathrm{L}(a,b)$ in the direction $\pi/2+\theta$.

The next proposition asserts that there is a one to one correspondence between length parameters $((\lambda_{1,\ell},\lambda_{1,r}),(\lambda_{2,\ell},\lambda_{2,r}),(\lambda_{3,\ell},\lambda_{3,r}))$ satisfying the train-track relations for $\pi_\ell=(1,3)$ and $\pi_r=(1,2)$ and parameters (a,b,θ) of the flat surface $r_{-\theta}\cdot\mathrm{L}(a,b)$ (up to rescaling).

Proposition 2. *Let $\pi_l=(1,2)(3)$, $\pi_r=(1,3)(2)$ and λ satisfy the train track relations (2). Then there exists a unique suspension datum ζ of (π,λ) such that the suspension $S(\pi,\zeta)$ is isomorphic, up to horizontal and vertical rescaling, to a surface of the form*

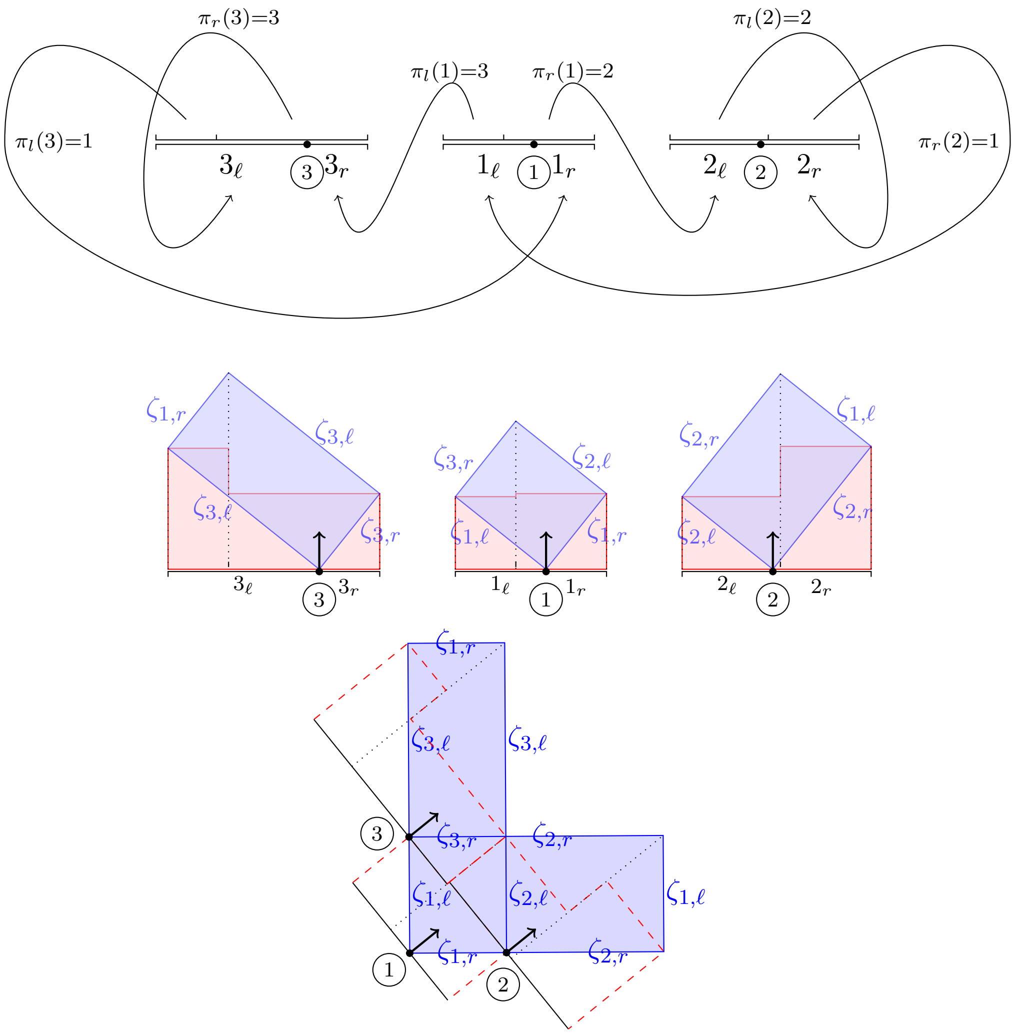


Figure 4: An interval exchange transformation with permutation datum $\pi_\ell = (1,3)(2)$, $\pi_r = (1,2)(3)$ and two views of the unique zippered rectangles associated to it which gives a translation surface of the form $r_\theta \cdot \mathrm{L}(a, b)$.

$r_\theta \cdot \mathrm{L}(a,b)$ with $\theta \in]0, \pi/2[$ endowed with its natural quadrangulation. Moreover, a, b and θ are deduced from the length-vector λ and by the following relations

$$\lambda_{1,\ell} = \lambda_{2,\ell} = -(1-b)\sin\theta \quad \lambda_{3,\ell} = -b\sin\theta \qquad \lambda_{1,r} = \lambda_{3,r} = (1-a)\cos\theta \quad \lambda_{2,r} = a\cos\theta.$$

Proof. For $i = 1, 2, 3$, the real parts $\zeta_{i,\ell}$ and $\zeta_{i,r}$ of the suspension datum ζ are respectively $\lambda_{i,\ell}$ and $\lambda_{i,r}$. We are looking for the imaginary parts such that the suspension $S(\pi,\zeta)$ is isomorphic to a translation surface of the form $r_\theta \cdot S(\pi,\zeta)$.

There are two independent equations on the imaginary parts imposed by the train track relations

$$\zeta_{1,\ell} = \zeta_{2,\ell} \qquad \text{and} \qquad \zeta_{1,r} = \zeta_{3,r}. \tag{3}$$

Furthermore, as in any surface $r_\theta \cdot \mathrm{L}(a,b)$ the sides of the quadrangulation are orthogonal, there are three other independent equations

$$\zeta_{i,\ell} \perp \zeta_{i,r} \quad \text{for } i = 1,2,3. \tag{4}$$

The equations (3) and (4) give 5 independent linear relations for our six parameters $\mathrm{Im}(\zeta_{i,m}), \mathrm{Im}(\zeta_{i,p})$. Hence there is exactly one solution up to rescaling. □

2.4 $\mathrm{L}(a,b)$ with extra symmetries: Calta-McMullen L's

Let X be a translation surface with singularities $\Sigma \subset X$. An *affine diffeomorphism* of X is an orientation preserving homeomorphism of X that permutes the singularities of the flat metric and acts affinely on the flat structure of X. We denote by $\mathrm{Aff}(X)$ the group of affine diffeomorphisms of $X \backslash \Sigma$. Let denote by $\Gamma(X)$ the image of the derivative map

$$d : \begin{cases} \mathrm{Aff}(X) & \rightarrow & \mathrm{SL}(2,\mathbb{R}) \\ f & \mapsto & df \end{cases}$$

which is called the *Veech group*. For surfaces in $\mathcal{H}(2)$, the affine group is isomorphic to the Veech group under the derivative map (see Proposition 4.4 in [HuLe06]). Next, we identify the Veech group and the affine group for surfaces in $\mathcal{H}(2)$.

A translation surface X is called a *Veech surface* (or *lattice surface*) if $\Gamma(X)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Veech surfaces were introduced in [Ve89] for dynamical purposes. Let X be a Veech surface. For an angle $\theta \in S^1$ if there exists an orbit of the linear flow which joins two singularities (*saddle connection*) then the linear flow $\phi_t^\theta : X \to X$ in direction θ is *parabolic*: any geodesic in direction θ is either a saddle connection or a loop and moreover there exists a non trivial element $\phi \in \mathrm{Aff}(X)$ which stabilizes all geodesics in that direction [Ve89]. The name parabolic comes from the fact that $d\phi \in \mathrm{SL}(2,\mathbb{R})$ is a parabolic matrix.

In the surface $\mathrm{L}(a,b) \in \mathcal{H}(2)$ the horizontal and vertical directions are *completely periodic*: all trajectories are either saddle connections or closed loops. A necessary condition for $\mathrm{L}(a,b)$ to be a Veech surface is that those two directions are parabolic, in other words there exists non identity matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ in $\Gamma(X)$. It turns out that these conditions are equivalent (which is a miracle of genus 2 translation surfaces).

Theorem 4 ([Ca04],[Mc03])**.** *The following conditions are equivalent:*

1. *the surface* $\mathrm{L}(a,b)$ *is a Veech surface,*
2. *horizontal and vertical directions in* $\mathrm{L}(a,b)$ *are parabolic,*
3. *either a and b are rational or there exists rational numbers x and y and a square-free integer $D > 1$ such that $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x) + y\sqrt{D}$.*

Now, we explicit the form of the parabolic elements in horizontal and vertical directions for parameters that satisfy condition 3 in the above Theorem. Those elements will be used to find paths in the Ferenczi-Zamboni induction. Let $0 < a < 1$ and $0 < b < 1$. Then, the stabilizers in $\mathrm{SL}(2,\mathbb{R})$ of, respectively, the bottom and top cylinders in the horizontal decomposition of $\mathrm{L}(a,b)$ are the two parabolic subgroups of $\mathrm{SL}(2,\mathbb{R})$ generated by

$$P_{bot} = \begin{pmatrix} 1 & 1/(1-b) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P_{top} = \begin{pmatrix} 1 & (1-a)/b \\ 0 & 1 \end{pmatrix}.$$

Remark that the matrix P_{bot} (resp. P_{top}) acts as a Dehn twist around the circumference of the bottom (resp. top) cylinder. The intersection $\langle P_{bot}\rangle \cap \langle P_{top}\rangle$ is non trivial (different from $\{1,-1\}$), if and only if there exist relatively prime positive integers m_h and n_h such that $m_h/(1-b) = n_h(1-a)/b$. The latter equation can be written as $m_h b = n_h(1-a)(1-b)$. By symmetry, the vertical direction is parabolic if and only if there exist relatively prime positive integers m_v and n_v such that $m_v a = n_v(1-a)(1-b)$. We call the 4-tuple (m_h, n_h, m_v, n_v) the *affine multi-twist parameters* of the Veech surface $\mathrm{L}(a,b)$. It is easy to show that the the existence of (m_h, n_h, m_v, n_v) for parameters a and b is equivalent to the third condition in Proposition 4 and more precisely

Proposition 3 ([Ca04],[Mc03]). *Let m_h, n_h, m_v and n_v be positive integers with m_h and n_h (resp. m_v and n_v) relatively primes. Then there exist unique real numbers a and b such that $0 < a < 1$, $0 < b < 1$ and $\mathrm{L}(a,b)$ is a Veech surface with affine multi-twists parameters (m_h, m_v, n_h, n_v). If we denote $\mu_h = n_h/m_h$ and $\mu_v = n_v/m_v$ then*

$$\begin{aligned}\frac{1}{1-a} &= \frac{1+(\mu_h-\mu_v)+\sqrt{1+(\mu_h-\mu_v)^2+2(\mu_h+\mu_v)}}{2}\\ \frac{1}{1-b} &= \frac{1+(\mu_v-\mu_h)+\sqrt{1+(\mu_h-\mu_v)^2+2(\mu_h+\mu_v)}}{2}\end{aligned}$$

In particular, $\mathrm{L}(a,a)$ is a Veech surface if and only if there exists $D = n/m \in \mathbb{Q}$ with

$$\frac{1}{1-a} = \frac{1+\sqrt{1+4D}}{2}.$$

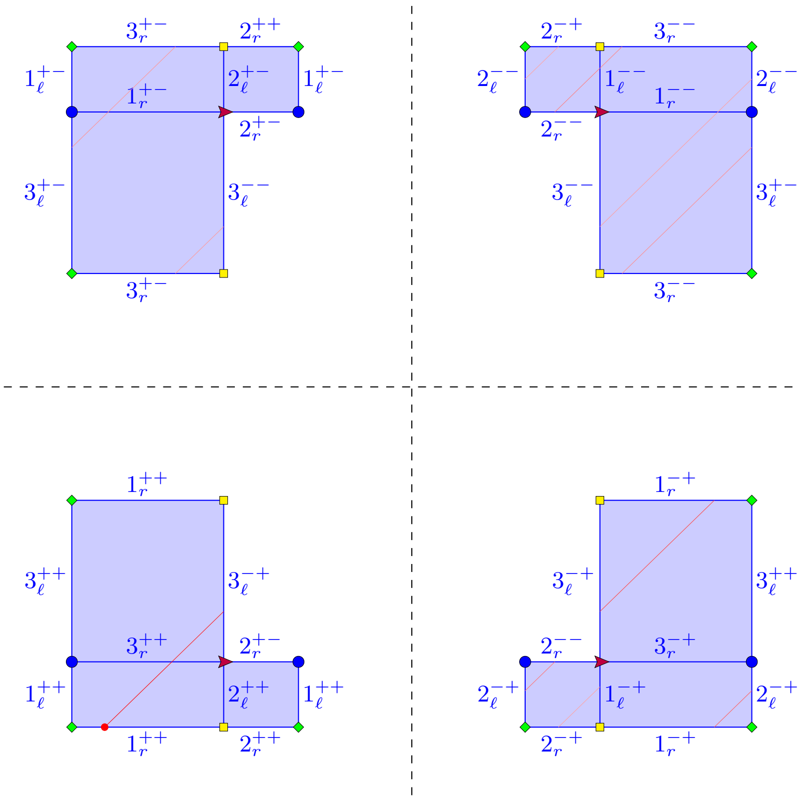
For a parameter a as above, the affine multi-twists parameters (m_h, m_v, n_h, n_v) of $\mathrm{L}(a,a)$ are $m_h = m_v = m$ and $n_h = n_v = n$.

2.5 From $\mathrm{L}(a,b)$ to $T(a,b)$: the wind-tree cocycle

Now, we describe a discretization of the billiard flow in $T(a,b)$ as a $\mathbb{Z}^2$-cocycle over an interval exchange transformation.

Given a rational billiard, there is a classical procedure to get a translation surface called *Katok-Zemliakov construction* or *unfolding procedure* (see the original articles [FoKe36] and [KaZm75] or the surveys [Ta95] or [MaTa02]). The unfolding procedure consists in taking reflected copies of the billiard instead of considering a reflected trajectory. The construction applies to the infinite billiard table $T(a,b)$ and is made of four copies associated to the four directions that a trajectory may take with a given initial angle. We denote by $\mathrm{X}_\infty(a,b)$ the translation surface obtained by unfolding the billiard table $T(a,b)$.

Proposition 4 ([DeHuLe]). *The infinite translation surface $\mathrm{X}_\infty(a,b)$ is a normal cover of $\mathrm{L}(a,b)$ and the Deck group $\mathrm{Deck}(\mathrm{X}_\infty(a,b), \mathrm{L}(a,b))$ is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes K$ where $K = \mathbb{Z}/2 \times \mathbb{Z}/2$ denotes the Klein four group. The intermediate quotient $\mathrm{X}_\infty(a,b)/\mathbb{Z}^2$ is a four fold cover of $\mathrm{L}(a,b)$ which corresponds to the unfolding of the billiard in a fundamental domain of $T(a,b)$ under the $\mathbb{Z}^2$-action.*



$1_r^{++}\ 3_r^{++}\ 3_l^{-+}\ 1_r^{-+}\ 2_l^{-+}\ 2_r^{--}\ 1_l^{--}\ 3_r^{--}\ 3_l^{+-}\ 1_r^{+-}\ 3_r^{+-}\ 3_l^{--}\ 1_r^{--}\ 2_l^{--}\ 2_r^{-+} \dots$

Figure 5: The lift in $\mathrm{X}(a,b)$ of the geodesic in $\mathrm{L}(a,b)$ from Figure 3.

The proof of the proposition is elementary and we refer to [DeHuLe]. Now, we consider how a geodesic in $\mathrm{X}_\infty(a,b)$ can be built from the ones in $\mathrm{L}(a,b)$. The Klein four group $K = \mathbb{Z}/2 \times \mathbb{Z}/2$ in the proposition naturally identifies with the group generated by the vertical and horizontal reflexions denoted respectively τ_v and τ_h.

In Section 2.2, we defined a symbolic coding of geodesics in $\mathrm{L}(a,b)$ on the alphabet $\mathcal{A} = \{1,2,3\} \times \{\ell, r\}$. The preimage of the quadrangulation of $\mathrm{L}(a,b)$ in $\mathrm{X}(a,b)$ gives a symbolic coding on the alphabet $\mathcal{A} \times K$. Given a geodesic γ (finite or infinite) in $\mathrm{L}(a,b)$ and its coding $w = (w_i)_i$ (in $\mathcal{A}^*$ or $\mathcal{A}^{\mathbb{N}}$) it has four lifts in $\mathrm{X}(a,b)$ and hence four possible codings. The *canonical* one that we denote $\tilde{w} = ((w_i, \kappa_i))_i$ is the one which starts in the copy labelled $id \in K$. Let $g : \mathcal{A} \to K$ be defined by

$$g(1_\ell) = g(1_r) = g(2_\ell) = g(3_r) = id \qquad g(2_r) = \tau_h \qquad \text{and} \qquad g(3_\ell) = \tau_v.$$

Then the canonical lift $\tilde{w} = ((w_i, \kappa_i))$ of w can be defined by

$$\kappa_0 = 1 \quad \text{and for } i \geq 0, \qquad \kappa_{i+1} = \kappa_i \, g(w_i).$$

The Klein four group K acts transitively on the four lifts of w. The three other cutting sequences for the lifts are $((a_i, \tau_v \kappa_i))_i$, $((a_i, \tau_h \kappa_i))_i$ and $((a_i, \tau_v \tau_h \kappa_i))_i$. In Figure 5 we give an example of the lift of a geodesic in $\mathrm{L}(a,b)$. To simplify notations we use $++$ (resp. $-+$, $+-$ and $--$) for the element $id \in K$ (resp. τ_v, τ_h and $\tau_v \tau_h$).

The surface $\mathrm{X}_\infty(a,b)$ is a $\mathbb{Z}^2$-cover of $\mathrm{X}(a,b)$. Hence, the preimage of the quadrangulation of $\mathrm{X}(a,b)$ determines a quadrangulation in $\mathrm{X}_\infty(a,b)$. We fix an origin in $\mathrm{X}_\infty(a,b)$ and consider a bijection between the faces of the quadrangulation and $\{1,2,3\}\times K\times\mathbb{Z}^2$. To lift the cutting sequence of a geodesic in $\mathrm{X}(a,b)$ to $\mathrm{X}_\infty(a,b)$ there is another cocycle which is defined on the copies $\{1,2,3\}\times\{id\}$ by

$$\begin{gathered} f((2_\ell,id)) = f((2_r,id)) = f((3_\ell,id)) = f((3_r,id)) = (0,0) \\ f((1_\ell,id)) = (1,0) \qquad \text{and} \qquad f((1_r,id)) = (0,1) \end{gathered}$$

and on the three other copies by the symmetry rule

$$\forall a \in \{1,2,3\}, \forall \kappa \in K,\ f((a,\kappa)) = \kappa \cdot f((a,id))$$

where τ_h and τ_v acts on $\mathbb{Z}^2$ by reflexion

$$\tau_h \cdot (x,y) = (x,-y) \qquad \tau_v \cdot (x,y) = (-x,y).$$

Proposition 5 ([DeHuLe])**.** *Let γ be a geodesic in $\mathrm{X}_\infty(a,b)$, w its cutting sequence on $\{1,2,3\}\times K\times\mathbb{Z}^2$ and $\overline{\gamma}$ its image in $T(a,b)$. Then for all $t>0$ we have*

$$\left| d(x,\phi_t^\theta(x)) - \|f^{(n)}(w)\|_2 \right| \leq \sqrt{2}$$

where $f^{(n)}(w) = f(w_0)f(w_1)\dots f(w_{n-1})$ and n is such that the geodesic from x has cut n sides of the quadrangulation before time t.

As the cover $\mathrm{X}_\infty(a,b)\to \mathrm{L}(a,b)$ is normal, we can build a non-commutative cocycle to lift the cutting sequences of geodesics in $\mathrm{X}(a,b)$ to $\mathrm{X}_\infty(a,b)$.

To simplify notations, we use a direct description from $\mathrm{L}(a,b)$ to $\mathrm{X}_\infty(a,b)$. Let $D_\infty = \mathbb{Z}\rtimes\mathbb{Z}/2$ be the infinite dihedral group (where $\mathbb{Z}/2$ acts by multiplication by -1 on $\mathbb{Z}$). We use the following notation for $G = D_\infty^2$. The generators of $K=\mathbb{Z}/2\times\mathbb{Z}/2$ are denoted by τ_v and τ_h. We use multiplicative notations and write the product rule in D_∞^2 as follows. For $(x_1,y_1),(x_2,y_2),(x,y)\in\mathbb{Z}^2$ we have

$$(x_1,y_1)(x_2,y_2) = (x_1+x_2,y_1+y_2) \qquad \tau_v(x,y) = (-x,y)\tau_v \qquad \tau_h(x,y) = (x,-y)\tau_h.$$

The cocycle which describe a cutting sequence in $\mathrm{X}_\infty(a,b)$ from one in $\mathrm{L}(a,b)$ is the map $f:\mathcal{A}\to G$ defined by

$$\begin{array}{lll} f(1_\ell) = (1,0) & f(2_\ell) = \tau_v & f(3_\ell) = (0,0) \\ f(1_r) = (0,1) & f(2_r) = (0,0) & f(3_r) = \tau_h. \end{array}$$

The cocycle f can be viewed as a function ϕ on the domain $E = E_1\sqcup E_2\sqcup E_3$ of an interval exchange transformation $T = T_{(\lambda,\pi)}$ with $\pi_l = (1,3)$ and $\pi_r = (1,2)$ which is constant on each interval $E_{i,\ell}$ (resp. $E_{i,r}$). Its value on $E_{i,\ell}$ is given by $f(i_\ell)$ (resp. by $f(i_r)$).

3 Divergent wind-tree cocycles

We recall the renormalization procedure of S. Ferenczi and L. Zamboni [FeZa, FeZa10] for linear flow on translation surfaces. The induction was used to obtain fine properties of interval exchanges in the hyperelliptic classes and many examples of exotic ergodic behaviors of interval exchange transformations. We use their induction to control the wind-tree cocycle. An other induction procedure for interval exchange transformations is the one of G. Rauzy [Ra79] which seems less adapted to our situation.

3.1 Ferenczi-Zamboni induction in hyperelliptic strata

We now recall the induction procedure introduced in [FeZa10]. In next sections, we restrict our study to the case of the stratum $\mathcal{H}(2)$ which is the subject of [FeZa] and corresponds to our surface $\mathrm{L}(a,b)$.

Let X be a translation surface with an admissible quadrangulation. The general principle of the Ferenczi-Zamboni induction consists in looking at a sequence of admissible quadrangulations of the surface such that the quadrilaterals become more and more flat in the direction of the linear flow. In the case of hyperelliptic strata, we consider only quadrangulations which are stable under the hyperelliptic involution. This restriction guarantees the existence of an induction procedure. The main point is that the hyperelliptic involution simplifies the train-track relations (2) (see Definition 2.4 and the discussion which follows in [FeZa10]).

Now, we describe the induction. Let $T = T_{\pi,\lambda} : E \to E$ be an interval exchange transformation on d intervals. We assume that $E = E_1 \sqcup \ldots \sqcup E_d$ is stable under the hyperelliptic involution. We want to define a new interval exchange transformation which corresponds to a first return map on a union of d subintervals $E' = E'_1 \sqcup \ldots \sqcup E'_d \subset E$ where for each $i = 1, \ldots, d$, $E'_i \subset E_i$ and E'_i contains the origin of E_i. For each i, we define its *state* (see Figure 6):

– i is in *left state* if $\lambda_{\pi_\ell(i),r} > \lambda_{i,\ell}$ (or equivalently $0 \in E_{i,l}$),

– i is in *right state* if $\lambda_{\pi_r(i),\ell} > \lambda_{i,r}$ (or equivalently $0 \in E_{i,r}$).

Knowing the state of each level we want to define T' by choosing among the following choices

– if i is in a left state either we choose $E'_i = E_i$ or $E'_i = E_{i,\ell}$,

– if i is in a right state either we choose $E'_i = E_i$ or $E'_i = E_{i,r}$.

Figure 6: Left or right state for the interval E_i of an interval exchange transformation.

Now, let $T = T_{\pi,\lambda}$ be an interval exchange transformation in a hyperelliptic strata such that it is stable under the hyperelliptic involution. As shown in [FeZa10] there are canonical choices which ensure that first of all the induction is always possible. Secondly, the induced interval exchange transformation is also stable under the hyperelliptic involution. The choice is done as follows. We consider the cycles in the disjoint cycle decomposition of the permutations π_ℓ and π_r. If there exists a cycle c of π_ℓ for which each element of c are in left state then we allow to perform a left induction for all of them. Such a cycle is called a *left branch of induction.* Formally a left induction step for c on $\pi = (\pi_\ell, \pi_r)$ and $\lambda = (\ell, r)$ gives the combinatorial data $\pi' = (\pi_\ell, \pi'_r)$ and $\lambda' = (\lambda_\ell, \lambda'_r)$ where

- for all i in c, $\lambda'_{i,r} = \lambda_{i,r} + \lambda_{\pi_r(i),\ell}$ and $\pi'_r(i) = \pi_r \circ \pi_\ell(i)$,
- for all i not in c, $\lambda'_{i,r} = \lambda_{i,r}$ and $\pi'_r(i) = \pi_r(i)$.

The definition of induction can easily be extended to suspensions by replacing $\lambda_{i,\ell}$ and $\lambda_{i,r}$ in the formulas above by respectively $\zeta_{i,\ell}$ and $\zeta_{i,r}$.

The following general theorem can be checked by hand for $\mathcal{H}(2)$ by building the so called "graph of graphs"(see Figure 7).

Theorem 5 ([FeZa10] Lemma 2.5 and Proposition 2.6)**.** *Let T be a hyperelliptic interval exchange in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ without saddle connection. Then, T admits at least one induction branch and any map induced from T by cutting all intervals in an induction branch is an hyperelliptic interval exchange transformation. Moreover, for any choice of infinite sequence of inductions for T such that*

- *if i is not in the induction branch at stage n then the state of i at step n and $n+1$ is the same*
- *each interval E_i for $i = 1, 2, \ldots, d$ is cut infinitely many times on its left and on its right.*

Conversely, given an infinite path of inductions in the graph of graphs starting from π that satisfies the two conditions above, there exists at least one parameter λ for which the interval exchange $T_{\pi,\lambda}$ has no saddle connection and from which we can perform these steps of induction.

We use the following multiplicative algorithm similar to the Gauss map for coding geodesics in the torus.

Definition 3. *The* multiplicative Ferenczi-Zamboni *algorithm on a symmetric interval exchange transformations is the algorithm which at odd steps performs all possible right inductions and even steps all possible left inductions.*

3.2 Description of the language in terms of induction

We now follow [FeZa] to describe the language of an interval exchange transformation in terms of one of its induction. Let $T = T_{\pi,\lambda}$ be a symmetric interval exchange transformation in $\mathcal{H}(2)$. For $i = 1, 2, 3$, we note $L_i = (\pi_l(i), r)$ and $R_i = (\pi_r(i), \ell)$ and $L = (L_1, L_2, L_3)$ and $R = (R_1, R_2, R_3)$. The words L_i and R_i are the two possible continuations of a letter of the form $(i, *)$.

Let c be a union of left (or a union of right) admissible branch of induction for T and let T' be the interval exchange transformation obtained from T by inducing with respect to c. Then the words L'_i and R'_i on this new interval exchange transformation seen from T can be described by the following rules

left induction on c	right induction on c
$L' = L$	$R' = R$
for i in c, $R'_i = L_i R_{\pi_\ell(i)}$	for i in c, $L'_i = R_i L_{\pi_r(i)}$
for i not in c, $R'_i = R_i$	for i not in c, $L'_i = L_i$

Starting from a symmetric interval exchange transformation and performing successively left and right inductions, we build a sequence of words $L_i^{(k)}$ and $R_i^{(k)}$. The possible inductions at each step are shown in Figure 7. From the definition of our multiplicative algorithm, at odd step $2k+1$ for $k \geq 0$, we perform a right induction and we have $R_i^{(2k+1)} = R_i^{(2k)}$. Similarly at even step $2k+k$ for $k \geq 0$, we perform left inductions and we have $L_i^{(2k+2)} = L_i^{(2k+1)}$.

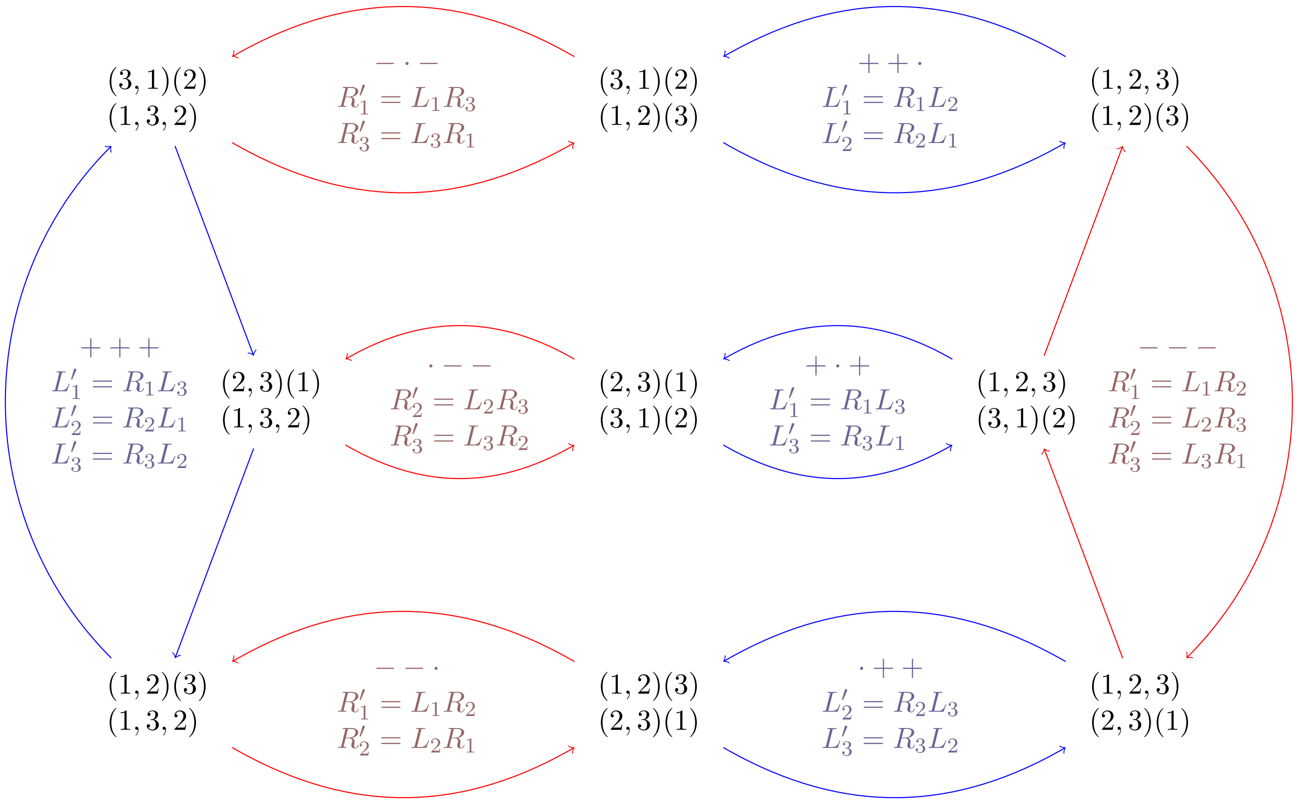


Figure 7: The states of the induction procedure in $\mathcal{H}(2)$ (left inductions are dashed). Some inductions which correspond to loops are missing. The light colored rectangle corresponds to the subset of inductions considered in Section 3.3.

3.3 A subset of parameters (a, b, θ) defined from the induction

We restrict our attention to a subset of inductions which simplifies considerably the form of the infinite words we obtain. This subset of possible inductions is similar to the one used in [FeZa] Section 5. For parameters (a, b, θ) associated to these inductions, we will be able to control the billiard orbits of $T(a, b)$ in direction θ.

Our graph of induction consists of the unique state $\pi = (\pi_\ell, \pi_r)$ with $\pi_\ell = (1, 3)$ and $\pi_r = (1, 2)$ which corresponds to the quadrangulation of $\mathrm{L}(a, b)$. We consider as induction steps

– the induction which are a succession of left inductions only that go from π to π,

– the induction which are a succession of right inductions only that go from π to π.

The subgraph of inductions is the part of the graph of graphs in the rectangle in Figure 7. There are two possible left induction from π associated respectively to the states $\ell{\cdot}\ell$ and ${\cdot}\ell{\cdot}$. As two left inductions commute, each step of the multiplicative algorithm (Definition 3) corresponds to a 2-tuple of integers (m, n) where m corresponds to the multiplicity of the loop of length two associated to state $\ell \cdot \ell$ and n is the multiplicity of the loop associated to the state ${\cdot}\ell{\cdot}$. A 2-tuple also encodes the left inductions. The induction algorithm is then a shift on sequences of two-tuples of integers $((m_k, n_k))_{k\geq 0}$.

The language of an interval exchange transformation that admits an induction which belongs to the subgraph is defined by two families of substitutions indexed by a 2-tuple (m, n) of integers

$$\sigma_\ell(m,n) = \begin{cases} L_1 \mapsto (R_1R_2)^m L_1 \\ L_2 \mapsto (R_2R_1)^m L_2 \\ L_3 \mapsto (R_3)^n L_3 \end{cases} \quad \text{and} \quad \sigma_r(m,n) = \begin{cases} R_1 \mapsto (L_1L_3)^m R_1 \\ R_2 \mapsto (L_2)^n R_2 \\ R_3 \mapsto (L_3L_1)^m R_3 \end{cases} \tag{5}$$

Because of the train track relations for π, namely $\lambda_{1,\ell} = \lambda_{2,\ell}$ and $\lambda_{1,r} = \lambda_{3,r}$, the possible vector-lengths are described by the 3-tuple $Z = (x_1, x_2, y_1, y_2) = (|\lambda_{2,\ell}|, |\lambda_{3,\ell}|, |\lambda_{3,r}|, |\lambda_{2,r}|) \in \mathbb{P}^3(\mathbb{R})$. The application of one step of the above algorithm corresponds, at the level of vector-lengths, to the application of one of the two projective maps below

$$F_\ell(m,n) : \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - m(y_1+y_2) \\ x_2 - ny_1 \\ y_1 \\ y_2 \end{pmatrix} \qquad F_r(m,n) : \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ y_1 - m(x_1+x_2) \\ y_2 - nx_1 \end{pmatrix}.$$

The multiplicative induction algorithm on the subgraph corresponds exactly to the map F defined in the introduction. We emphasize that not all vector-lengths parameters $Z = (\lambda_{2,\ell}, \lambda_{3,\ell}, \lambda_{3,r}, \lambda_{2,r})$ admit a continued fraction expansion with respect to this algorithm (the domain is a Cantor set). We recall that we name *F-renormalizable* a quadruple of length parameters Z for which the induction is exactly prescribed by our subgraph with one vertex at $\pi = ((1,3),(1,2))$. From Theorem 5, an F-renormalizable 3-tuple determines a unique sequence of 2-tuples $((m_k, n_k))_{k\geq 0}$ such that

– for each $k \geq 1$ either $m_k \neq 0$ or $n_k \neq 0$,

– if $m_k = 0$, then $m_{k+1} \neq 0$ and $n_{k+1} = 0$

– for infinitely many i, $m_{2k} \neq 0$ (resp. $m_{2k+1} \neq 0$),

– for infinitely many i, $n_{2k} \neq 0$ (resp. $n_{2k+1} \neq 0$).

Reciprocally, from Proposition 5, we know that every sequence of 2-tuples of integers that satisfy the above conditions gives a F-renormalizable vector-lengths.

3.4 Renormalizable slopes in Veech $\mathrm{L}(a, b)$

In this section, to a Veech surface of the form $\mathrm{L}(a, b)$ (see Proposition 4 for a characterization) we build a set of slopes $\Lambda \subset S^1$ for which the corresponding vector lengths of the interval exchange transformation F-renormalizable.

Proposition 6. *Let* $\mathrm{L}(a,b)$ *be a Veech surface and* (m_h, n_h, m_v, n_v) *its Dehn multi-twist parameters (see Section 2.4). Let* (s_h, s_v) *be the widths of the associated parabolic matrices in the Veech group. Then for* $\theta \in (0, \pi/2)$ *of the form*

$$\tan(\theta)^{-1} = a_0\, s_h + \cfrac{1}{a_1\, s_v + \cfrac{1}{a_2\, s_h + \cfrac{1}{a_3\, s_v + \cfrac{1}{\dots}}}}$$

the interval exchange transformation T *associated to* (a, b, θ) *by Proposition 2 is* F*-renormalizable. The convergents associated to the restricted multiplicative Ferenczi-Zamboni induction of* T *are* $(a_{2k} m_h, a_{2k} n_h)$ *for even* k *and* $(a_{2k+1} m_v, a_{2k+1} n_v)$ *for odd* k.

Proof. It is more convenient to consider coordinates in $\mathbb{P}^1(\mathbb{R})$ instead of $\theta \in (0, \pi/2)$. To the angle θ we associate the (oriented) slope $x = \tan(\theta)^{-1} = (\cos(\theta) : \sin(\theta)) \in \mathbb{P}^1(\mathbb{R})$.

Let ρ_ℓ (resp. ρ_r) be the vertical (resp. the horizontal) parabolic element which stabilizes the Veech surface $\mathrm{L}(a,b)$. We note

$$\rho_\ell = \begin{pmatrix} 1 & 0 \\ s_v & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_r = \begin{pmatrix} 1 & s_h \\ 0 & 1 \end{pmatrix}.$$

As the Veech group of $\mathrm{L}(a,b)$ is a lattice $s_h s_v \geq 1$, otherwise the group generated by ρ_ℓ and ρ_r won't be discrete. In particular, if $x = x(\theta)$ has the form given in the statement, then the sequence $(a_k)_k$ associated to $x \in \mathbb{P}^1(\mathbb{R})$ is unique. More precisely, the expansion of x is defined from a modified continued fraction algorithm. Let $\psi_\ell :]0, 1/s_v[\to]1/s_v, \infty[$ and $\psi_r :]s_h, \infty[\to]0, s_h[$ be the two maps

$$\psi_\ell(x) = \frac{1}{1/x - \left\lfloor \frac{1/x}{s_v} \right\rfloor s_v} \qquad \text{and} \qquad \psi_r(x) = x - \left\lfloor \frac{x}{s_h} \right\rfloor s_h$$

As $s_h s_v \geq 1$ the domain of ψ_ℓ and ψ_r are disjoint and we define ψ to be the map that equals ψ_ℓ on $(0, 1/s_v)$ and ψ_r on (s_h, ∞). The map ψ is associated to the shift on the sequence $(a_k)_k$ that defines $x = \tan(\theta)$: if x is defined by the sequence $(a_0, a_1, a_2, \ldots)$ then $\psi(x)$ is defined either by the sequence $(0, a_1, a_2, \ldots)$ if $a_0 \neq 0$ or $(a_2, a_3, \ldots)$ if $a_0 = 0$.

The maps ψ_ℓ and ψ_r correspond to the standard projective action of powers of the inverses of two matrices ρ_ℓ and ρ_r that corresponds to the horizontal and vertical multi-twist in $\mathrm{L}(a,b)$:

Let $\theta \in S^1$ be such that $x = \tan(\theta)^{-1}$ admits an infinite expansion with respect to ψ and $S(\pi, \zeta)$ be the suspension associated to $r_{\pi/2-\theta} \cdot \mathrm{L}(a,b)$ as in Proposition 2. Assume that $a_0 \neq 0$, then we can perform a left induction with parameters $(a_0 m_h, a_0 n_h)$. Let $S(\pi, \zeta')$ be the surface obtained after this step of left induction. Then the vertical direction in $S(\pi, \zeta')$ corresponds to the direction x in $\rho_h^{a_0} \cdot \mathrm{L}(a,b)$ or equivalently, to the direction $\psi(x)$ in $\mathrm{L}(a,b)$. □

We now estimate the Hausdorff dimension of the set of renormalizable slope in a Veech surface of the form $\mathrm{L}(a,b)$.

Proposition 7. *Let s and t be positive real numbers with $st \geq 1$ and consider $R(s,t) \subset \mathbb{R}$ the set of real numbers x of the form*

$$x = a_0\, s + \cfrac{1}{a_1\, t + \cfrac{1}{a_2\, s + \cfrac{1}{a_3\, t + \cfrac{1}{\cdots}}}}$$

where $a_0 \geq 0$ and $a_k \geq 1$ for $k \geq 1$ are integers. Then

- *for any $\lambda > 0$, $\mathrm{Hdim}(R(s_h, s_v)) = \mathrm{Hdim}(R(\lambda s, \lambda^{-1} t))$,*
- *the map $s \mapsto \mathrm{Hdim}(R(s,s))$ is decreasing and not smaller than $1/2$.*

Proof. We use the following notation

$$[a_0, a_1, \ldots]_{s,t} = a_0\, s + \cfrac{1}{a_1\, t + \cfrac{1}{a_2\, s + \cfrac{1}{a_3\, t + \cfrac{1}{\cdots}}}}.$$

The map

$$\begin{array}{ccc} R(s,t) & \to & R(\lambda s, \lambda^{-1} t) \\ x = [a_0, a_1, \ldots]_{s,t} & \mapsto & [a_0, a_1, \ldots]_{\lambda s, \lambda^{-1} t} \end{array}$$

is just a multiplication by λ. As bi-Lipschitz map preserves Hausdorff dimension, $\mathrm{Hdim}\, R(s,t) = \mathrm{Hdim}\, R(\lambda s, \lambda^{-1} t)$.

The fact that $s \mapsto \mathrm{Hdim}\, R(s,s)$ is decreasing is immediate from the construction. In the following, we fix $s > 1$ and show that the Hausdorff dimension of $R(s,s)$ is not smaller than $1/2$. It follows from [He] Chapter 9 that the Hausdorff dimension can be computed with the canonical covers. More precisely, for a tuple $v = (a_1, \ldots, a_k)$ we define the s-convergents as follows

$$\begin{array}{lll} p_0 = 1 & p_1 = s\, a_2 & p_{k+1} = s\, a_k p_k + p_{k-1} \\ q_0 = s\, a_1 & q_1 = s^2\, a_2 a_1 + 1 & q_{k+1} = s\, a_k q_k + q_{k-1}. \end{array}$$

Then the quantity $[a_1, a_2 \ldots, a_k]_{s,s} = p_k/q_k$ converges to the real number

$$x = \cfrac{1}{a_1\, s + \cfrac{1}{a_2\, s + \cfrac{1}{a_3\, s + \cfrac{1}{\cdots}}}}.$$

Let $\mathbf{v} = (a_1, \ldots, a_k)$ be a k-tuple of integers. We denote by $|\mathbf{v}|_s$ the denominator q_k of the continued fraction $[s\, a_1, \ldots, s\, a_k]$ constructed above. To define the Hausdorff dimension we first define the function λ as follows

$$\lambda_s(\sigma) = \sup \left\{ \lambda > 0; \limsup_{r \to \infty} \lambda^{-r} \sum_{\mathbf{v} \in \mathbb{N}^r} |\mathbf{v}|_s^{-\sigma} \right\}$$

where $\mathbb{N}$ is the set of positive integers. The length of the interval $\{[a_1,\ldots,a_k+t]_s; t\in[0,1]\}$ is $|(a_1,\ldots,a_k)|_s^{-1}|(a_1,\ldots,a_k+1)|_s^{-1}$. Hence, the quantity $|\mathbf{v}|_s^{-\sigma}$ in the definition of λ is up to a factor 2, the length of an interval of the canonical cover at step r to the power $\sigma/2$. The Hausdorff dimension of $R(s,s)$ is

$$\mathrm{Hdim}\, R(s,s) = \frac{1}{2}\inf\{\sigma; \lambda_s(\sigma) < 1\}.$$

But $\lambda_s(1) = 0$ as the serie $\sum_{\mathbf{v}\in\mathbb{N}^r} |\mathbf{v}|_s^{-1}$ diverges for any s. The Hausdorff dimension of $R(s,s)$ is then not smaller than $1/2$. □

Corollary 1. *For any Veech* $\mathrm{L}(a,b)$ *with parabolics* ρ_ℓ *and* ρ_r *in respectively vertical and horizontal direction, the set of slopes renormalizable by* ρ_ℓ^m *and* ρ_r^n *has positive Hausdorff measure bounded below by* $1/2$ *for any positive integer* m *and* n.

3.5 Divergent cocycles: proof of the main theorem

This section is devoted to the proof of Theorem 2. In next section, we illustrate all computations with the simple example of $L(1/2,1/2)$ with slope $\theta = \arctan(\sqrt{2}-1)$ which corresponds to the periodic expansion $((1,2),(1,2),\ldots)$.

We first describe the strategy of the proof. Let T be an interval exchange transformation without saddle connection that is F-renormalizable. Then, for each step $k\in\mathbb{N}$, the k-th step of the Ferenczi-Zamboni induction can be used to decompose the coding of a trajectory with the six pieces $L_i^{(k)}$ and $R_i^{(k)}$ for $i=1,2,3$. The size of the pieces grows with k and more precisely, the pieces at step $k+1$ are concatenations of the pieces at step k. The rule to glue the pieces is given by the substitutions defined in Section 3.2 and depends on the convergents $((m_k,n_k))_{k\geq 0}$ of the restricted Ferenczi-Zamboni induction of T. Now assume that T satisfies the statement of Theorem 2. Then we prove that for each $k\geq 0$ the wind-tree cocycle over T has no "local self-intersection". More precisely, for $i=1,2,3$ and $k\geq 0$, let $\mathcal{L}_i^{(k)}$ and $\mathcal{R}_i^{(k)}$ be the subsets of $G = D_\infty\times D_\infty$ made of the values taken by the wind-tree cocycle on the finite pieces $L_i^{(k)}$ and $R_i^{(k)}$. In the cutting sequence considered as concatenation of pieces $L_i^{(k)}$ and $R_i^{(k)}$, the values taken by the cocycles on all pieces are translates of $\mathcal{L}_i^{(k)}$ and $\mathcal{R}_i^{(k)}$ by an element of $G = (\mathbb{Z}\rtimes\mathbb{Z}/2)^2$. We prove that for $k>2$, the values of level $k+1$ are built in such way that each part from level k do not intersect each other. The reason why we need $k>2$ is due to the fact that for step 1 (resp. step 2) the trajectory can rebound between two vertical scatterers (resp. horizontal scatterers) which implies that the values of the cocycle during this period take only two values (x,y) and $(x,y)\tau_v$ (resp. (x,y) and $(x,y)\tau_h$). In particular we prove the stronger statement that the trajectory in the wind-tree model are "self-avoiding".

We fix for the remaining of the section a triple (a,b,θ) that fulfill the hypothesis of Theorem 2 and consider the associated interval exchange transformation T. We denote by $((m_k,n_k))_{k\geq 0}$ the convergents of the restricted Ferenczi-Zamboni induction of T and $(L^{(k)},R^{(k)})$ the 6-tuple of words that describe the coding of the orbits in T. For $i=1,2,3$, let $\mathcal{L}_i^{(k)}$ and $\mathcal{R}_i^{(k)}$ the subsets of $G = D_\infty\times D_\infty$ made of values taken by the wind-tree cocycle on respectively $L_i^{(k)}$ and $R_i^{(k)}$. At step $k=0$

$$\left(L^{(0)},R^{(0)}\right) = ((3_r,2_r,1_r),(2_\ell,1_\ell,3_\ell)). \tag{6}$$

and hence

$$\begin{array}{lll} \mathcal{L}_1^{(0)} = \{(0,0)\} & \mathcal{L}_2^{(0)} = \{(0,0),(0,0)\tau_h\} & \mathcal{L}_3^{(0)} = \{(0,0),(0,1)\} \\ \mathcal{R}_1^{(0)} = \{(0,0)\} & \mathcal{R}_2^{(0)} = \{(0,0),(1,0)\} & \mathcal{R}_3^{(0)} = \{(0,0),(0,0)\tau_v\}. \end{array} \tag{7}$$

The six words $L_i^{(k)}$ and $R_i^{(k)}$ for $i = 1,2,3$ are defined recursively by the substitutions σ_ℓ and σ_r in (5). More precisely denoting $L = L^{(k-1)}$, $L' = L^{(k)}$, $R = R^{(k-1)}$ and $R' = R^{(k)}$ we have

$$\begin{array}{l|l} \text{for odd steps } k & \text{for even steps } k \\ \hline R' = R & L' = L \\ L_1' = (R_1R_2)^{m_k}\ L_1 & R_1' = (L_1L_3)^{m_k}\ R_1 \\ L_2' = (R_2R_1)^{m_k}\ L_2 & R_2' = (L_2)^{n_k}\ R_2 \\ L_3' = (R_3)^{n_k}\ L_3 & R_3' = (L_3L_1)^{m_k}\ R_3 \end{array} \tag{8}$$

In order to simplify notations, we use the above notations in many proofs: R' and R (resp. L' and L) for $R^{(k+1)}$ and $R^{(k)}$ (resp. for $L^{(k+1)}$ and $L^{(k)}$).

The first step of the proof consists in analyzing the value of the cocycle at the endpoints of each of the pieces $L^{(k)}$ and $R^{(k)}$. In Lemma 1, we prove that the endpoints are always oriented in the same way for all k, more precisely the value of the cocycle g with value in K is constant. Then, using this property, we prove Lemma 2 which gives an explicit values for theses endpoints.

As it was defined in Section 2.5, the wind-tree cocycle f decomposes into two parts. The first one g with values in $K = \mathbb{Z}/2 \times \mathbb{Z}/2$ and the other one with values in $\mathbb{Z}^2$. Let $g : \mathcal{A}^* \to K$ be the composition of f with the projection $G \to K$. There is a natural lift of K into G and we set for $w \in \mathcal{A}^*$, $\overline{f}(w) = f(w)g(w) \in \mathbb{Z}^2$.

Lemma 1. *We have for any* $k \geq 0$

$$\begin{array}{lll} g\left(L_1^{(k)}\right) = g(3_r) = 1 & g\left(L_2^{(k)}\right) = g(2_r) = \tau_h & g\left(L_3^{(k)}\right) = g(1_r) = 1 \\ g\left(R_1^{(k)}\right) = g(2_\ell) = 1 & g\left(R_2^{(k)}\right) = g(1_\ell) = 1 & g\left(R_3^{(k)}\right) = g(3_\ell) = \tau_v \end{array}.$$

and

$$\overline{f}\left(L_1^{(k)}\right) = \overline{f}\left(L_2^{(k)}\right) \in \mathbb{N} \times \mathbb{N} \quad \text{and} \quad \overline{f}\left(R_1^{(k)}\right) = \overline{f}\left(R_3^{(k)}\right) \in \mathbb{N} \times \mathbb{N}$$

and

$$\overline{f}\left(L_3^{(k)}\right) \in \{0\} \times \mathbb{N} \quad \text{and} \quad \overline{f}\left(R_2^{(k)}\right) \in \mathbb{N} \times \{0\}.$$

Proof. The statement is true for $k = 0$ from the definition of $\left(L^{(0)}, R^{(0)}\right)$ in (6) and definition of the wind-tree cocycle. Then we proceed by induction. We do the proof for odd steps, the case of even steps being similar. Assume that k is even and that $L = L^{(k-1)}$ and $R = R^{(k-1)}$ satisfies the conclusion of the lemma. Let $L' = L^{(k)}$ and $R' = R^{(k)}$. As $R' = R$ the conclusion holds for R'. From the definition of L' we have

$$f(L_1') = f((R_1R_2)^{m_k}\ L_1) = f(R_1R_2)^{m_k}\ f(L_1).$$

From induction hypothesis, all of $f(R_1)$, $f(R_2)$ and $f(L_1)$ belongs to $\mathbb{N}^2$. Hence $f(L_1') \in \mathbb{N}$. Similarly for L_2' we have

$$f(L_2') = f(R_2R_1)^{m_k}\ f(L_2).$$

From induction hypothesis, $f(L_2)$ is of the form $(h,v)\tau_h$ with $h,v \in \mathbb{N}$ and hence get the conclusion for L_2'. Now consider the case of L_3'. As $n_k = 2n'$ is even (assumption in theorem 2) one has

$$f(L_3') = (f(R_3)^2)^{n'}\, f(L_3).$$

But as $f(R_3)$ is of the form $(h,v)\tau_v$ with $(h,v) \in \mathbb{N}$, we have $f(R_3)^2 = (0,2v)$. We hence get the conclusion for R_3'. This ends the proof of the lemma. $\square$

Now, we compute explicitly the sequence $\overline{f}\left(L_j^{(k)}\right), \overline{f}\left(R_j^{(k)}\right)$ which from Lemma 1 depends only on six parameters. Let $X^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)$ and $Y^{(k)} = \left(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}\right)$ be the vectors with non negative integer entries such that

$$\begin{array}{ll} \overline{f}\left(L_1^{(k)}\right) = \overline{f}\left(L_2^{(k)}\right) = \left(x_1^{(k)}, x_2^{(k)}\right) & \overline{f}\left(L_3^{(k)}\right) = \left(0, x_3^{(k)}\right) \\ \overline{f}\left(R_1^{(k)}\right) = \overline{f}\left(R_3^{(k)}\right) = \left(y_2^{(k)}, y_1^{(k)}\right) & \overline{f}\left(R_2^{(k)}\right) = \left(y_3^{(k)}, 0\right) \end{array} \tag{9}$$

Our convention for y_1 and y_2 may seem strange but is explained by the nice formula in the lemma below.

Lemma 2. *For odd steps, only the coordinates of X are modified as*

$$X^{(2k+1)} = X^{(2k)} + \begin{pmatrix} 0 & m_{2k+1} & m_{2k+1} \\ m_{2k+1} & 0 & 0 \\ n_{2k+1} & 0 & 0 \end{pmatrix} Y^{(2k)}.$$

For even steps, only Y is modified as

$$Y^{(2k)} = Y^{(2k-1)} + \begin{pmatrix} 0 & m_{2k} & m_{2k} \\ m_{2k} & 0 & 0 \\ n_{2k} & 0 & 0 \end{pmatrix} X^{(2k-1)}.$$

Proof. We omit the proof which proceeds by induction and follows the one of Lemma 1. $\square$

We now build explicit "boxes" around the trajectory. More precisely we find the minimum and maximum values of each coordinates of the sets $\mathcal{L}_i^{(k)}$ and $\mathcal{R}_i^{(k)}$. In order to take care of the horizontal excursions of $L_3^{(k)}$ and vertical excursions of $R_2^{(k)}$, we add one coordinate to the vectors X' and Y'. Let $x_4^{(0)} = y_4^{(0)} = 1$ and define recursively for odd steps

$$x_4^{(2k+1)} = \begin{cases} \max\left(x_4^{(2k)}, y_2^{(2k)}\right) & \text{if } n_{2k+1} \neq 0, \\ x_4^{(2k)} & \text{otherwise,} \end{cases} \qquad \text{and} \qquad y_4^{(2k+1)} = y_4^{(2k)} \tag{10}$$

and for even step

$$x_4^{(2k)} = x_4^{(2k-1)} \qquad \text{and} \qquad y_4^{(2k)} = \begin{cases} \max\left(y_4^{(2k-1)}, x_2^{(2k-1)}\right) & \text{if } n_{2k} \neq 0, \\ y_4^{(2k-1)} & \text{otherwise} \end{cases} \tag{11}$$

We first start by the formal definition of a box.

Definition 4. *Let $\pi_h : \mathbb{Z}^2 \to \mathbb{Z}$ (resp. $\pi_v : \mathbb{Z}^2 \to \mathbb{Z}$) be the projection on the first (resp. the second) coordinate. Let $A \subset \mathbb{Z}^2$. The* box *of A is the 4-tuple*

$$\mathrm{Box}(A) = (\min \pi_h(A), \min \pi_v(A), \max \pi_h(A), \max \pi_v(A)) \in \mathbb{Z}^4.$$

By extension, we call for $i = 1, 2, 3$ the *box* of the word $L_i^{(k)}$ (resp. $R_i^{(k)}$) the box of the subset $\mathcal{L}_i^{(k)}$ (resp. $\mathcal{R}_i^{(k)}$). The boxes around the pieces $(L^{(k)}, R^{(k)})$ is given in the following lemma.

Lemma 3. *We have*

$$\begin{array}{c}
\mathrm{Box}\left(L_1^{(k)}\right) = \mathrm{Box}\left(L_2^{(k)}\right) = \left(0, 0, x_1^{(k)}, x_2^{(k)}\right) \qquad ,\\
\mathrm{Box}\left(R_1^{(k)}\right) = \mathrm{Box}\left(R_3^{(k)}\right) = \left(0, 0, y_2^{(k)}, y_1^{(k)}\right) \qquad ,\\
\mathrm{Box}\left(L_3^{(2i)}\right) = \left(0, 0, x_4^{(k)}, x_3^{(k)}\right), \qquad \mathrm{Box}\left(R_2^{(2i)}\right) = \left(0, 0, y_3^{(k)}, y_4^{(k)}\right).
\end{array}$$

Proof. The cases of $L_1^{(k)}$, $L_2^{(k)}$ is straightforward from the proof of Lemma 1 as well as the case of $R_1^{(k)}$ and $R_3^{(k)}$. We prove by induction the formula for the box of $L_3^{(k)}$, the case of $L_3^{(k)}$ being similar. There is nothing to prove at even steps as $L_3^{(2k)} = L_3^{(2k-1)}$. Assume that the conclusion of the lemma holds at an even step $k-1$ and denote $L = L^{(k-1)}$ and $L' = L^{(k)}$. We recall that

$$L_3' = (R_3)^{n_k} \ L_3$$

where n_k is an even number by assumption in Theorem 2. If $n_k = 0$, then $L_3' = L_3$, and from (2) we have $x_3' = x_3$ and from (10) $x_4' = x_4$. Hence the box fits in this case. Now assume that $n_k \neq 0$. We know from our induction hypothesis that $\mathrm{Box}(R_3) = (0, 0, y_2, y_1)$. As the word R_3 ends with τ_v, for any $n \geq 1$ we have $\mathrm{Box}\left((R_3)^n\right) = (0, 0, y_2, n\, y_1)$. From Lemma 2 we have $f\,(R_3')^{(n_k)} = (0, n_k\, y_1) \in \{0\} \times \mathbb{N}$ and $f\,(L_3') = (0, x_3)$. The word L_3' is the concatenation of $(R_3)^{n_k}$ and L_3 and is hence contained in the box with bottom-left corner $(0, 0)$ and up-right corner $(\max\left(x_4, y_2\right), n_k\, y_1 + x_3) = (x_4', x_3')$. ☐

The following lemma states that the trajectories in the billiard are self-avoiding at large scales. In other words if the trajectory crosses at time t_0 and t_1 then the difference $|t_1 - t_0|$ should be small. In the proof of Theorem 2 below, we refine the argument to prove that at small scales intersection does not appear as well.

Lemma 4. *Let $k \geq 2$ be such that all entries of $X^{(k-1)}$ and $Y^{(k-1)}$ are positive. Let W' be one of the six words $L_i^{(k)}$ or $R_i^{(k)}$ for $i = 1, 2, 3$ and $W' = W_1 W_2 \dots W_p$ its decomposition given by the Ferenczi-Zamboni induction where each W_j equals one of the six words $L_i^{(k-1)}$ or $R_i^{(k-1)}$ for $i = 1, 2, 3$. We denote by $\mathcal{W}_j$ the set of values taken by the wind-tree cocycle on W_j. Let j and j' be two distinct elements of $\{1, \dots, p\}$. Then the subsets $f(W_1 \dots W_{j-1})\ \mathcal{W}_j$ and $f(W_1 \dots W_{j'-1})\ \mathcal{W}_{j'}$ are disjoint if $|j - j'| \neq 1$ and have only one intersection point otherwise.*

Proof. The decomposition of W' depends on the parity of k and is given by the rules (8). We do the proof at an odd step of the induction. Let $W' = L_1' = (R_1 R_2)^{m_k} L_1 = W_1 W_2 \dots W_{2m_k+1}$ where as in the proof of the preceding lemmas $L_i' = L_i^{(k)}$, $R_i' = R_i^{(k)}$, $L_i = L_i^{(k-1)}$ and $R_i = R_i^{(k-1)}$. From lemmas 2 and 3, we know that the values of the wind-tree cocycle on R_1 and R_2 respectively ends in the top-right corner and the bottom-right corner of the respective box $\mathrm{Box}(R_1)$ and $\mathrm{Box}(R_2)$. From Lemma 1, we deduce that in the word $W' = (R_1 R_2)^{m_k} L_1$ each of the individual box $f(W_1 \ldots W_{j-1})\ \mathcal{W}_j$ is glued to the preceding at only one point which is $f(W_1 \ldots W_{j-1})$. The proof works the same for $L_2' = (R_2 R_1)^{m_k} L_2$ and $L_3' = (R_3)^{n_k} L_3$.

Because of our assumption on the entries of $X^{(k-1)}$ and $Y^{(k-1)}$, the intersection of two non consecutive boxes is empty. □

We are now ready to prove our main theorem.

Proof of Theorem 2. Let $w \in \mathcal{A}^{\mathbb{Z}}$ be a cutting-sequence of an orbit of the interval exchange determined by (a, b, θ) which satisfies the hypothesis of Theorem 2. Then, for each k, we decompose w as a word on the alphabet $\mathcal{A}^{(k)} = \{L_1^{(k)}, L_2^{(k)}, L_3^{(k)}, R_1^{(k)}, R_2^{(k)}, R_3^{(k)}\}$. We stress that the origin of w (the letter in position 0) should be shifted in order to have a decomposition on $\mathcal{A}^{(k)}$ which starts at position 0. But this does not matter for our purpose and consider that w has no origin when it is decomposed with respect to the alphabet $\mathcal{A}^{(k)}$. From Lemma 4, we know that for k big enough, the pieces of size k contained in a piece of size $k+1$ are disjoint except for one possible value which occurs at the end of a box and the begining of the next one. Hence, if the box that contains the origin grows arbitrarily on the left and on the right, the trajectory determined by w is divergent. But if the box does not growth arbitrarily on the right, say it is blocked at N, then it means that in the future the orbit of p encounters a singularity of the interval exchange transformation after N steps.

We now refine the argument at small scales to prove that the trajectory in the infinite billiard is self-avoiding. It follows from the combinatorics of the surface $\mathrm{L}(a, b)$, that two consecutive pieces of level k may intersect in only few cases (see Figure 3) which corresponds to block on which the cocycle remains constant:

- either in the word $w = 3_r 3_\ell^n$ where $n \geq 0$,
- or in the form $w = 2_\ell 2_r^n$ where $n \geq 0$.

In the first case (resp. the second) the word is followed by 1_r (resp. 1_ℓ). In the billiard table, the cutting sequence w lift to a piece of trajectory which reflects between two horizontally (resp. vertically) consecutive scatterers but does not reflect vertically (resp. horizontally). In particular, the trajectory is self-avoiding in w. □

4 An example

We now consider the example of the periodic sequence of convergents $((1, 2), (1, 2), \ldots)$ associated to the square tiled surface $\mathrm{L}(1/2, 1/2)$ and the slope $\sqrt{2} - 1$. For this slope, the Ferenczi-Zamboni induction is periodic or in other words the interval exchange transformation is self-similar. The Figure 8 shows a three level of boxes for an associated orbit in the infinite billiard table $T(1/2, 1/2)$.

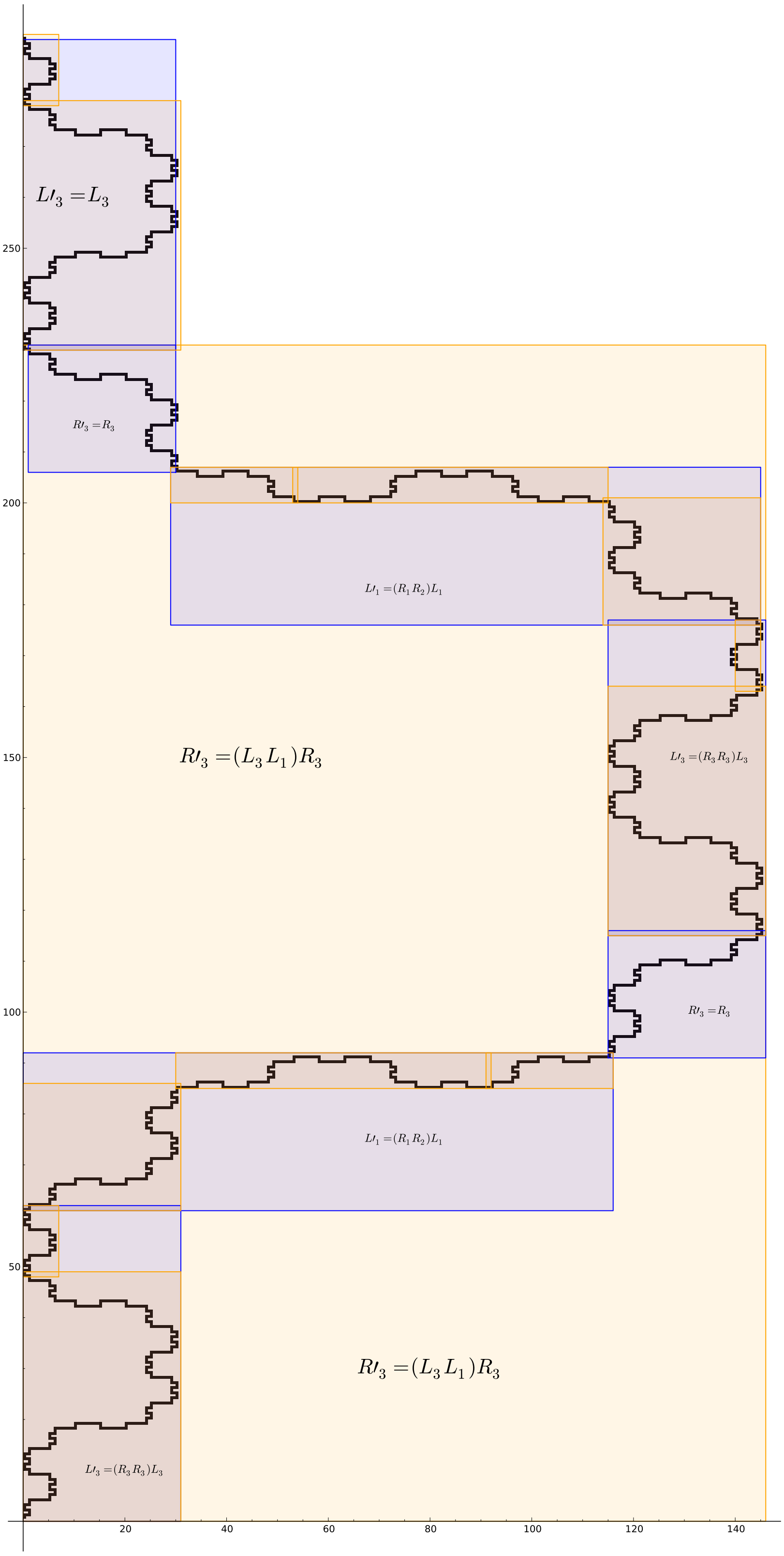


Figure 8: Three levels of boxes for the wind-tree cocycle over L(1/2, 1/2) and the slope $\sqrt{2} - 1$. The line in black corresponds to the quadrilaterals visited by an orbit in the wind-tree model $T(1/2, 1/2)$.

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Annexe B

Diffusion du vent dans les arbres

Dans cette deuxième annexe, nous reproduisons une version de l'article en collaboration avec P. Hubert et S. Lelièvre sur la diffusion du vent dans les arbres. La numérotation des pages tout au long de l'article suit celle de l'article et non de la thèse. Ainsi, la page 81 de cette thèse est numérotée 1. Cette numérotation spéciale sera en italique.

Diffusion for the periodic wind-tree model

joint work with Pascal Hubert and Samuel Lelièvre

Abstract

The periodic wind-tree model is an infinite billiard in the plane with identical rectangular scatterers disposed at each integer point. We prove that independently of the size of the scatterers, generically with respect to the angle, the polynomial diffusion rate in this billiard is 2/3.

Résumé

Diffusion du vent dans les arbres

Le vent dans les arbres périodique est un billard infini construit de la manière suivante. On considère le plan dans lequel sont placés des obstacles rectangulaires identiques à chaque point entier. Une particule (identifiée à un point) se déplace en ligne droite (le vent) et rebondit de manière élastique sur les obstacles (les arbres). Nous prouvons qu'indépendamment de la taille des obstacles et génériquement par rapport à l'angle initial de la particule le coefficient de diffusion polynomial des orbites de ce billard est 2/3.

1 Introduction

The wind-tree model is a billiard in the plane introduced by P. Ehrenfest and T. Ehrenfest in 1912 ([EhEh]). We study the periodic version studied by J. Hardy and J. Weber [HaWe]. A point moves in the plane $\mathbb{R}^2$ and bounces elastically off rectangular scatterers following the usual law of reflection. The scatterers are translates of the rectangle $[0,a] \times [0,b]$ where $0 < a < 1$ and $0 < b < 1$, one centered at each point of $\mathbb{Z}^2$. We denote the complement of obstacles in the plane by $\mathrm{T}(a,b)$ and refer to it as the *wind-tree model* or the *infinite billiard table*. Our aim is to understand dynamical properties of the wind-tree model. We denote by $\phi_t^\theta : \mathrm{T}(a,b) \to \mathrm{T}(a,b)$ the billiard flow: for a point $p \in \mathrm{T}(a,b)$, the point $\phi_t^\theta(p)$ is the position of a particle after time t starting from position p in direction θ.

It is proved in [HaWe] that the rate of diffusion in the periodic wind-tree model is $\log t \log\log t$ for very specific directions (generalized diagonals which corresponds to angles of the form $\arctan(p/q)$ with $p/q \in \mathbb{Q}$). Their result was recently completed by J.-P. Conze and E. Gutkin [CG] who explicit the ergodic decomposition of the billiard flow for those directions. K. Frączek and C. Ulcigrai recently proved that generically the billiard flow is non-ergodic. P. Hubert, S. Lelièvre and S. Troubetzkoy [HLT] proved that for a residual set of parameters a and b, for almost every direction θ, the flow in direction θ is recurrent. In this paper, we compute the polynomial rate of diffusion of the orbits which is valid for almost every direction θ. We get the following result which is independent of the size of the scatterers.

Theorem 1. *Let $d(.,.)$ be the Euclidean distance on $\mathbb{R}^2$.*

1. *If a and b are rational numbers or can be written as $1/(1-a) = x+y\sqrt{D}$, $1/(1-b) = (1-x)+y\sqrt{D}$ with $x, y \in \mathbb{Q}$ and D a positive square-free integer then for Lebesgue-almost all θ and every point p in $\mathrm{T}(a,b)$ (with an infinite forward orbit)*

$$\limsup_{T\to+\infty} \frac{\log d(p, \phi_T^\theta(p))}{\log T} = \frac{2}{3}.$$

2. *For Lebesgue-almost all $(a,b) \in (0,1)^2$, Lebesgue-almost all θ and every point p in $\mathrm{T}(a,b)$ (with an infinite forward orbit)*

$$\limsup_{T\to+\infty} \frac{\log d(p, \phi_T^\theta(p))}{\log T} = \frac{2}{3}.$$

The conclusion of the first statement is stronger but holds for specific parameters. We do not know if the result holds for *every* parameter $(a,b) \in (0,1)^2$. A classification of invariant measures for the action of Borel subgroup of $\mathrm{SL}(2,\mathbb{R})$ on the stratum of Abelian differentials $\mathcal{H}(2)$ would certainly be a first step in this direction.

By the $\mathbb{Z}^2$-periodicity of the billiard table $\mathrm{T}(a,b)$, our problem reduces to understand deviations of a-$\mathbb{Z}^2$ cocycle over the billiard in a fundamental domain. On the other hand, as the barriers are horizontals and verticals, an orbit in $\mathrm{T}(a,b)$ with initial angle θ from the horizontal takes at most four different directions $\{\theta, \pi-\theta, -\theta, \pi+\theta\}$ (the billiard is rational). A standard construction consisting of unfolding the trajectories [Ta], called the Katok-Zemliakov construction, the billiard flow can be replaced by a linear flow on a (non compact) translation surface which is made of four copies of $\mathrm{T}(a,b)$ that we denote $\mathrm{X}_\infty(a,b)$. The surface $\mathrm{X}_\infty(a,b)$ is $\mathbb{Z}^2$-periodic and we denote $\mathrm{X}(a,b)$ the quotient of $\mathrm{X}_\infty(a,b)$ under the $\mathbb{Z}^2$ action. As, the unfolding procedure of the billiard flow is equivariant with respect to the $\mathbb{Z}^2$ action, $\mathrm{X}(a,b)$ can be also be seen as the unfolding of the billiard in a fundamental domain of the action of $\mathbb{Z}^2$ on the billiard table $\mathrm{T}(a,b)$.

The position of the particle in $\mathrm{X}_\infty(a,b)$ can be tracked from $\mathrm{X}(a,b)$. More precisely, the position of the particle starting from $p \in \mathrm{X}_\infty(a,b)$ in direction θ can be approximated by the intersection of a geodesic in $\mathrm{X}(a,b)$ with a cocycle $f \in H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ describing the infinite cover $\mathrm{X}_\infty(a,b) \to \mathrm{X}(a,b)$. The growth of intersection of geodesics with cocycle in a translation surface is equivalent to the growth of certain Birkhoff sums over an interval exchange transformation. The estimation can be obtained from the action of $\mathrm{SL}(2,\mathbb{R})$ on strata of translation surfaces $\mathcal{H}_g(\alpha)$ and more precisely to the Teichmüller flow which corresponds to the action of diagonal matrices $g_t = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t}\end{smallmatrix}\right)$ (see Section 2 for precise definitions). Proved by A. Zorich [Zo1, Zo2] the Kontsevich-Zorich cocycle over the Teichmüller flow can be used to estimate the deviations of Birkhoff sums for generic interval exchange transformations with respect to the Lebesgue measure. More precisely, he proved that the Lyapunov exponents of the Kontsevich-Zorich cocycle measures the polynomial rate of deviations. G. Forni [Fo] relates this phenomenon to the obstruction of cohomological equations and extends Zorich's proof to a more general context.

The surface $\mathrm{X}(a,b)$ is a covering of the genus 2 surface $\mathrm{L}(a,b)$ which is a so called L-shaped surface that belongs to the stratum $\mathcal{H}(2)$. The orbit of $\mathrm{X}(a,b)$ for the Teichmüller flow belongs to a sub-locus of the moduli space $\mathcal{H}(2^4)$ that we call $\mathcal{G}$. The classification of $\mathrm{SL}(2,\mathbb{R})$-ergodic measures for the locus $\mathcal{G}$ follows from the fundamental work of C. McMullen [Mc1, Mc2, Mc3] for the stratum $\mathcal{H}(2)$. He proved that the only $\mathrm{SL}_2(\mathbb{R})$ invariant submanifolds in $\mathcal{H}(2)$ are the Teichmüller curves (which corresponds to case 1 in Theorem 1) and the stratum itself (case 2). The only $\mathrm{SL}_2(\mathbb{R})$ invariant probability measures are the Lebesgue measures on these loci. To prove Theorem 1 we use asymptotic theorems with respect to those measures.

We now formulate a generalization of A. Zorich's and G. Forni's theorems about deviations of ergodic averages that is a central step in the proof of Theorem 1. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differential and $Y \in \mathcal{H}(\alpha)$ a translation surface. The Teichmüller flow (g_t) can be used to renormalize the trajectories of the linear flow on Y. The Kontsevich-Zorich cocycle $B^{(t)}(Y) : H^1(Y;\mathbb{R}) \to H^1(Y;\mathbb{R})$ (or KZ cocycle) measures the growth of cohomology vectors along the Teichmüller geodesic $(g_t \cdot Y)_t$. Let μ be a g_t-invariant ergodic probability measure on $\mathcal{H}(\alpha)$. It follows from [Fo], that the KZ cocycle is integrable for the measure μ. From Oseledets multiplicative ergodic theorem, there exists real numbers $\nu_1 > \nu_2 > \ldots > \nu_k > 0$, such that for μ-almost every non zero Abelian differential $Y \in \mathcal{H}(\alpha)$ there exists a unique flag

$$H^1(Y;\mathbb{R}) = F_1^u \supset F_2^u \supset \ldots \supset F_k^u \supset F_{k+1}^u = F^c \supset F_k^s \supset \ldots \supset F_1^s \supset F_0^s = \{0\}$$

such that for any norm $\|.\|$ on $H^1(Y;\mathbb{R})$

1. if $f \in F_i^u \backslash F_{i+1}^u$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(\omega)\cdot f\|}{\log t} = \nu_i,$$

2. if $f \in F_i^s \backslash F_{i-1}^s$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(\omega)\cdot f\|}{\log t} = -\nu_i,$$
3. if $f \in F^c \backslash F_k^s$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(\omega)\cdot f\|}{\log t} = 0.$$

There exists also positive integers m_i for $i = 1, \ldots, k$ and an integer m such that for μ almost all Abelian differentials ω the filtration satisfied

- the dimension of F_i^s is $m_1 + \ldots + m_i$,
- the dimension of F^c is $m_1 + \ldots + m_k + 2m$,
- the dimension of F_i^u is $m_1 + \ldots + m_{i-1} + 2m_i + \ldots + 2m_k + 2m$.

From the definition of the Teichmüller flow and the KZ cocycle, it follows that $\nu_1 = 1$. Forni proved that $m_1 = 1$ [Fo]. The *Lyapunov spectrum of the KZ* cocycle is the multiset of numbers

$$\nu_1 = 1 \quad \underbrace{\nu_2 \ldots \nu_2}_{m_2 \text{ times}} \quad \ldots \quad \underbrace{\nu_k \ldots \nu_k}_{m_k \text{ times}} \quad \underbrace{0 \ldots 0}_{2m \text{ times}} \quad \underbrace{-\nu_k \ldots -\nu_k}_{m_k \text{ times}} \quad \ldots \quad \underbrace{-\nu_2 \ldots -\nu_2}_{m_2 \text{ times}} \quad -1 = -\nu_1$$

The numbers ν_i for $i = 1, \ldots, k$ are called the *positive Lyapunov exponents*. The subspace $F^s = F_k^s$ is called the *stable space* of the KZ cocycle.

Theorem 2. *Let μ be a g_t-ergodic measure on a stratum of Abelian differentials. Let X be a translation surface in the support of μ. Let ν_i for $i = 1, \ldots, k$ denotes the positive Lyapunov exponents of the KZ cocycle and $F_i^u(Y)$, $F^c(Y)$ and $F_i^s(Y)$ the components of the flag of the Oseledets decomposition for an Oseledets generic surface Y.*

For μ-almost every translation surface $Y \in \mathcal{H}(\alpha)$ Oseledets generic, for every point $p \in Y$ with an infinite forward orbit

1. *along the unstable space the growth is polynomial: for all $f \in F_i^u \backslash F_{i+1}^u$*
$$\forall i \in \{1, \ldots, k\},\ \limsup_{T\to\infty} \frac{\log |\langle f, \gamma_T(p)\rangle|}{\log T} = \nu_i,$$
2. *along the central space the growth is sub-polynomial: for all $f \in F^c \backslash F_k^s$*
$$\limsup_{T\to\infty} \frac{\log |\langle f, \gamma_T(p)\rangle|}{\log T} = 0,$$
3. *along the stable space the growth is bounded: there exists a constant C such that for all $f \in F^s$*
$$\forall T \geq 0,\ |\langle f, \gamma_T(p)\rangle| \leq C\|f\|.$$

Theorem 2 has first been proved by A. Zorich [Zo0, Zo1, Zo2] for the Lebesgue measure on a connected component of a stratum or equivalently for a generic interval exchange transformation. G. Forni [Fo] extended the theorem for a very large class of functions and for certain measures. More precisely, his proof of the lower bound relies on the existence of a particular translation surface in the support of the measure. A. Bufetov [Bu] gave a proof of case 1 of Theorem 2 (when the cocycle f is associated with a positive

Lyapunov exponent) in the general context of symbolic dynamics which applies in particular to translation flows (Proposition 2. and 5. of [Bu]). Our approach uses Veech's zippered rectangle [Ve1] and gives a concrete version of the renormalization process by the Teichmüller flow and the Kontsevich-Zorich cocycle in the flavor of [Zo1, Zo2] and [Fo].

On the other hand, from results of A. Eskin, M. Kontsevich and A. Zorich [EKZ2] about sum of Lyapunov exponents in hyperelliptic loci, we deduce that the Lyapunov exponent for $\mathrm{X}(a,b)$ which controls the deviation in the wind-tree model equals $2/3$. The value $2/3$ comes from algebraic geometry. More precisely, it corresponds to the degree of a subbundle of the Hodge bundle over the moduli space of complex curves (or Riemann surfaces) in which the wind-tree cocycle belongs to.

The paper is organised as follows. In Section 2 we introduce the tools from Teichmüller theory which are involved in our proof of Theorem 1. In Section 3, we detail the unfolding procedure and prove that the distance in Theorem 1 corresponds to an intersection of a geodesic in $\mathrm{X}(a,b)$ with an integer cocycle. Then we reformulate Theorem 1 in the language of translation surfaces (see Theorem 6). In Section 4 we compute the Lyapunov exponents relative to every measure on $\mathcal{H}(2^4)$ which is supported on the closure of the $\mathrm{SL}(2,\mathbb{R})$-orbit of a surfaces $\mathrm{X}(a,b)$. Section 5 is devoted to the proof of Theorem 2. Finally, in Section 6 we prove how the generic results for surfaces in $\mathcal{H}(2^4)$ can be transfered to results on the specific surfaces $\mathrm{X}(a,b)$ which form a set of zero measure in $\mathcal{H}(2^4)$ with respect to any measure we are interested in.

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Contents

2 Background

The basic objects in this paper are several flavours of flat surfaces:

– closed compact translation surfaces – equivalently, closed compact Riemann surfaces endowed with a holomorphic 1-form;

– infinite-area periodic translation surfaces.

For general references on translation surfaces and interval exchange transformations we refer the reader to the survey of A. Zorich [Zo3], J.-C. Yoccoz [Yo] or the notes of M. Viana [Vi].

A *translation surface* is a surface which can be obtained by edge-to-edge gluing of polygons in the plane using translations only. Such a surface is endowed with a flat metric (the one from $\mathbb{R}^2$) and canonic directions. There is a one to one correspondence between compact translation surfaces and compact Riemann surfaces equipped with a non-zero holomorphic 1–form. If (Y, ω) is a Riemann surface together with a holomorphic one-form, the flat metric corresponds to $|\omega|^2$. In particular, the area of (Y, ω) is $i/2 \int \omega \wedge \overline{\omega}$.

In a translation surface, directions are globally defined. Hence the geodesic flow in a direction can be defined on the surface. There is a canonic vertical direction in each translation surface and we refer to the flow in this direction as the *linear flow*. The flow in the direction $\theta \in [0, 2\pi)$ for the differential ω on Y is the linear flow of $e^{-i\theta}\omega$ on Y.

The moduli space of translation surfaces of genus g, denoted $\mathcal{H}_g$, is a (complex g-dimensional) vector bundle over the moduli space of Riemann surfaces $\mathcal{M}_g$. Moduli spaces $\mathcal{H}_g$ decompose into strata according to the degrees of zeros of the corresponding 1-forms. If $\alpha = (\alpha_1, ..., \alpha_s)$ is a partition of the even number $2g - 2$, $\mathcal{H}_g(\alpha)$ or simply $\mathcal{H}(\alpha)$ denotes the stratum consisting of 1–forms with zeros of degrees α_1, ..., α_s, on genus g-Riemann surfaces. These strata can have up to three connected components, which were classified by M. Kontsevich and A. Zorich [KZ], and distinguished by two invariants: hyperellipticity and parity of spin structure.

We denote by $\mathcal{H}^{(1)}(\alpha) \subset \mathcal{H}(\alpha)$ the codimension 1 subspace which consists of area 1 translation surfaces.

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on components of strata $\mathcal{H}(\alpha)$ coming from the linear action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$. More precisely, let (Y, ω) be a translation surface obtained by gluing a finite family of polygons (P_i) and $g \in \mathrm{SL}_2(\mathbb{R})$. Then the surface $g \cdot (Y, \omega)$ is the surface obtained by gluing the polygons $(g \cdot P_i)$. The *Teichmüller geodesic flow* on $\mathcal{H}_g$ is the action of the diagonal matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The image of the orbits $(g_t \cdot (X, \omega))_t$ in $\mathcal{M}_g$ are geodesic with respect to the Teichmüller metric. Each stratum $\mathcal{H}_g(\alpha)$ carries a natural *Lebesgue measure*, invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. Moreover, this action preserves the area and hence $\mathcal{H}^{(1)}(\alpha)$. H. Masur [Ma] and independently W. Veech [Ve1] proved that on each component of a normalised stratum $\mathcal{H}^{(1)}(\alpha)$ the total mass of the Lebesgue measure is finite and the geodesic flow acts ergodically with respect to this measure. Another important one parameter flow on $\mathcal{H}(\alpha)$ is the *horocycle flow* given by the action of $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

More generally, one can consider the strata of quadratic differentials with at most simple poles $\mathcal{Q}_g(\alpha)$ where α is an integer partition of $4g - 4$. The degree α_i corresponds to a conic point of angle $(2 + \alpha_i)\pi$. A translation surface associated to a quadratic differential may has non trivial holonomy with value in $\{1, -1\}$. The action of $\mathrm{SL}(2, \mathbb{R})$ on Abelian differentials extends to quadratic differentials.

Stabilisers for the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}_g$ or $\mathcal{Q}_g$, called *Veech groups*, are discrete non-cocompact subgroups of $\mathrm{SL}_2(\mathbb{R})$; they are trivial (i.e. either $\{\mathrm{Id}\}$ or $\{\mathrm{Id}, -\mathrm{Id}\}$) for almost every surface in each stratum component, and in exceptional cases are lattices (i.e. finite-covolume subgroups) in $\mathrm{SL}_2(\mathbb{R})$. In such cases, the surface satisfies the Veech dichotomy: in every direction, the linear flow is either uniquely ergodic, or decomposes the surface into a finite union of cylinders of periodic trajectories (see [Ve1]). Closed compact translation surfaces with a lattice Veech group are exactly those whose $\mathrm{SL}_2(\mathbb{R})$-orbit is closed in the corresponding stratum component. They are called *Veech surfaces*. Their orbits project to *Teichmüller curves* in the moduli space $\mathcal{M}_g$ of closed compact Riemann surfaces of genus g. A translation surface is a *square-tiled surface* if it is a ramified cover of the torus $\mathbb{R}^2/\mathbb{Z}^2$ with only 0 as ramification point. Square-tiled surfaces are examples of Veech surfaces. Their Veech groups are commensurable to $\mathrm{SL}_2(\mathbb{Z})$.

The simplest stratum besides the one of tori is $\mathcal{H}(2)$ which consists of equivalence classes of 1–forms with a double zero (in flat surfaces terms a cone point of angle 6π) on Riemann surfaces of genus two. Important examples of such surfaces are given by the family of surfaces $\mathrm{L}(a, b)$ with $0 < a < 1, 0 < b < 1$ which is built as follows (see also Figure 1). Let $0 < a < 1$ and $0 < b < 1$. Consider the polygon with extremal points $(0, 0)$, $(1 - a, 0)$, $(1, 0)$, $(1, 1 - b)$, $(1 - a, 1 - b)$, $(1 - a, 1)$, $(0, 1)$, $(0, 1 - b)$ and glue the opposite sides together:

1. $[(0, 0), (1 - a, 0)]$ with $[(0, 1), (1 - a, 1)]$ (the side h_1 labeled on Figure 1),
2. $[(1 - a, 0), (1, 0)]$ with $[(1 - a, 1 - b), (1, 1 - b)]$ (the side h_2),
3. $[(0, 0), (0, 1 - b)]$ with $[(1, 0), (1, 1 - b)]$ (the side v_1),
4. $[(0, 1 - b), (0, 1)]$ with $[(1 - a, 1 - b), (1 - a, 1)]$ (the side v_2).

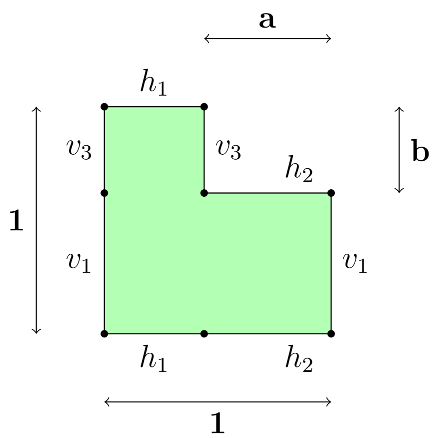


Figure 1: The surface $\mathrm{L}(a, b)$ built from a L-shaped polygon.

The stratum $\mathcal{H}(2)$ is connected and is the best understood. It was proven that the Teichmüller curves are generated by surfaces of the form $\mathrm{L}(a, b)$.

Theorem 3 (Calta [Ca], McMullen [Mc1, Mc2])**.** *The surface* $\mathrm{L}(a, b)$ *is a Veech surface if and only if*

1. either $a, b \in \mathbb{Q}$ in which case $\mathrm{L}(a,b)$ is square-tiled,

2. or there exists $x, y \in \mathbb{Q}$ and $D > 1$ a square-free integer such that $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x) + y\sqrt{D}$.

Moreover, any Teichmüller curve in $\mathcal{H}(2)$ contains (up to rescaling the area) a surface of the form $\mathrm{L}(a,b)$.

In his fundamental work, C. McMullen [Mc3] proved a complete classification theorem for $\mathrm{SL}_2(\mathbb{R})$-invariant measures and closed invariant set.

Theorem 4 (McMullen, [Mc3] Theorems 10.1 and 10.2 p. 440–441)**.** *The only $\mathrm{SL}(2,\mathbb{R})$-invariant irreducible closed subsets of $\mathcal{H}(2)$ are the Teichmüller curves and the whole stratum. The only $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures are the Haar measure carried on Teichmüller curves and the Lebesgue measure on the stratum.*

Let $g \geq 2$. The *Hodge bundle* E_g is the real vector bundle of dimension $2g$ over $\mathcal{M}_g$ where the fiber over $X \in \mathcal{M}_g$ is the real cohomology $H^1(X;\mathbb{R})$. Each fibre $H^1(X;\mathbb{R})$ has a natural lattice $H^1(X;\mathbb{Z})$ which allows identification of nearby fibers and definition of the Gauss-Manin (flat) connection. The holonomy along the Teichmüller geodesic flow provides a symplectic cocycle called the *Kontsevich-Zorich cocycle*. For each g_t-invariant ergodic probability measure for the Teichmüller geodesic flow on $\mathcal{H}_g$, this cocycle has associated Lyapunov exponents. Based on computer experimentations, M. Kontsevich [KZ0] conjectured a formula for the sum of positive Lyapunov exponents of the cocycle for Lebesgue measures on strata as well as for Veech surfaces. These formula are now fully proven [EKZ1, EKZ2].

In some concrete situations, the existence of automorphisms provides an $\mathrm{SL}_2(\mathbb{R})$-equivariant splitting of the Hodge bundle. Under suitable assumptions for the $\mathrm{SL}(2,\mathbb{R})$-subbundles (relative to variations of Hodge structure), it appears that for each of them there is a formula for the sum of positive Lyapunov exponents of the restricted Kontsevich-Zorich cocycle. Sometimes even individual Lyapunov exponents can be computed (see [BM], [FMZ], [EKZ1]). For the hyperelliptic locii of a stratum, the sum of positive Lyapunov exponents does not depend on the $\mathrm{SL}(2,\mathbb{R})$-ergodic measure.

Theorem 5 (Eskin-Kontsevich-Zorich [EKZ2], Corollary 1 p. 14)**.** *Let μ be an $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measure on a stratum $\mathcal{H}_g(\alpha)$ of Abelian differential. Assume that μ comes from the orientation covering morphism of a $\mathrm{SL}(2,\mathbb{R})$-invariant (regular) measure $\overline{\mu}$ on a stratum of quadratic differentials on the sphere $\mathcal{Q}(d_1, d_2, \ldots, d_n)$. Then, the sum of positive Lyapunov exponents $\nu_1 \geq \ldots \geq \nu_g$ for the measure μ is given by*

$$\nu_1 + \ldots + \nu_g = \frac{1}{4} \sum_{j \text{ with } d_j \text{ odd}} \frac{1}{d_j + 2}.$$

In particular the value of the sum does not depend on the measure but only on the stratum $\mathcal{Q}(d_1, d_2, \ldots, d_n)$. For the condition of *regular measure* which appears in the statement of Theorem 5 we refer to Definition 1 p. 9 of [EKZ2]. We emphasise that all known $\mathrm{SL}(2,\mathbb{R})$-ergodic measures on strata of Abelian differentials are regular.

For infinite-area translation surfaces, it is not clear what the good notions of moduli spaces are. However, the action of $\mathrm{SL}_2(\mathbb{R})$ still makes sense, and Veech groups can be defined [Va1, Va2]. An *infinite periodic translation surface* is an infinite area translation surface which is an infinite normal cover of a (finite area) translation surface. We say Γ-infinite translation surface to specify the Deck group Γ. Examples of $\mathbb{Z}$-infinite translation surfaces are studied by P. Hubert and G. Schmithüsen in [HS] and a general formalism is introduced by P. Hooper and B. Weiss in [HoWe]. For some particularly symmetric examples, it is possible to get a very complete picture of the dynamics [HoHuWe]. The family of surfaces $\mathrm{X}_\infty(a, b)$ obtained by unfolding the billiard tables $T(a, b)$ are $\mathbb{Z}^2$-infinite translation surfaces.

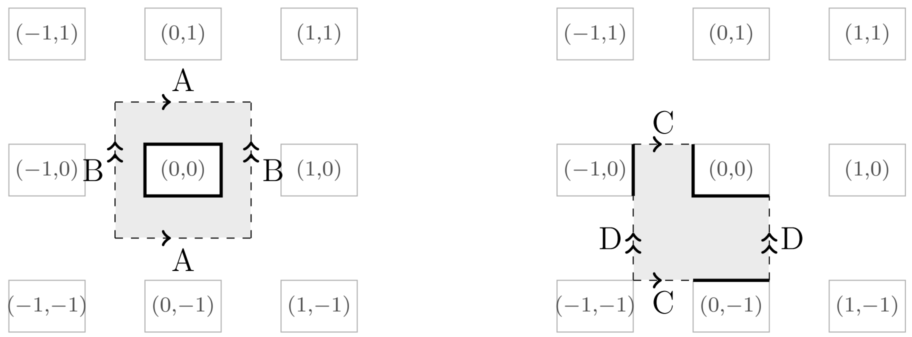
3 From infinite billiard table to finite surface

First of all, the flow in the billiard table $\mathrm{T}(a,b)$ is invariant under $\mathbb{Z}^2$ translation. Secondly, the angles between the scatterers are multiples of $\pi/2$ and the Katok-Zemliakov construction conjugates the billiard flow on $T(a,b)$ to a linear flow on an infinite translation surface $\mathrm{X}_\infty(a,b)$. Using these two ingredients, we reduce the study of the billiard flow into the study of a $\mathbb{Z}^2$-cocycle over the liner flow of a finite translation surface $\mathrm{X}(a,b)$. The surface $\mathrm{X}(a,b)$ obtained by unfolding a fundamental domain of the table $\mathrm{T}(a,b)$ is an intermediate cover between the finite surface $\mathrm{L}(a,b)$ of genus 2 and the infinite surface $\mathrm{X}_\infty(a,b)$. The surface $\mathrm{X}(a,b)$ is the main actor of this paper.

Notation: For the whole section, we fix $0 < a < 1$ and $0 < b < 1$.

3.1 Unfolding the fundamental domain

A fundamental domain for the $\mathbb{Z}^2$ action on the infinite billiard $T(a,b)$ can be seen either as a torus with a square obstacle inside (see Figure 2a) or as a surface $\mathrm{L} = \mathrm{L}(a,b)$ with barriers on its boundary (see Figure 2b). The Katok-Zemliakov construction (or



(a) Fundamental domain of the billiard table $\mathrm{T}(a,b)$ as a torus with a rectangle scatterer.

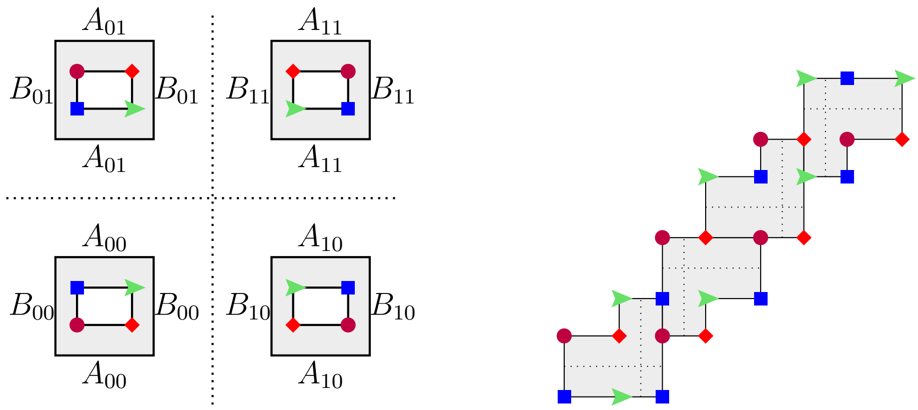
(b) Fundamental domain of the billiard table $\mathrm{T}(a,b)$ as a L shaped surface with barriers.

Figure 2: Two versions of the fundamental domains for the billiard table $\mathrm{T}(a,b)$. The boundaries of the scatterers are thick and the arrows together with letters indicate the gluings.

unfolding procedure) of the billiard in the fundamental domain gives a surface $\mathrm{X}(a,b)$ made of 4 reflected copies of the fundamental domain (see Figure 3). The surface $\mathrm{X}(a,b)$ was studied in the particular case $a = b = 1/2$ by different authors [LS], [S], [FMZ], [EKZ1] and is called in this particular case the 6-escalator (see Figure 3b for the origin of the name).

Lemma 1. *The surface* $\mathrm{X}(a,b)$ *is a genus* 5 *surface in* $\mathcal{H}(2^4)$*. It is a normal unramified cover of the surface* $\mathrm{L}(a,b)$ *with a Deck group* K *isomorphic to the Klein four-group* $K = \mathbb{Z}/2 \times \mathbb{Z}/2$*.*

Proof. The billiard table $\mathrm{T}(a,b)$ is invariant under horizontal and vertical reflections as well as the billiard in a fundamental domain. It is then straightforward to show that $\mathrm{X}(a,b)$ is an unramified normal cover of $\mathrm{L}(a,b)$ with group $\mathbb{Z}/2 \times \mathbb{Z}/2$. A direct computation shows that $\mathrm{X}(a,b)$ has 4 singularities of angle 6π (see Figure 3). $\square$



(a) Unfolding the toric fundamental domain of Figure 2a.

(b) Unfolding the L shaped fundamental domain of Figure 2b.

Figure 3: Two versions of the surface $\mathrm{X}(a, b)$ obtained by unfolding the billiard in a fundamental domain. The gluings of edges are indicated by labels in case of ambiguity.

3.2 The surface $\mathrm{X}_\infty(a, b)$ as a $\mathbb{Z}^2$ cover of $\mathrm{X}(a, b)$

As we did for unfolding the fundamental domain of the infinite billiard, we consider the unfolding of the whole billiard table $\mathrm{T}(a, b)$. The unfolding leads to a non compact surface that we denote $\mathrm{X}_\infty(a, b)$ which is made of four copies of the initial billiard. As the unfolding commutes with the action of $\mathbb{Z}^2$ the surface $\mathrm{X}(a, b)$ is also the $\mathbb{Z}^2$ quotient of $\mathrm{X}_\infty(a, b)$. We use this description to rewrite the distance in Theorem 1 as an intersection of a geodesic segment in $\mathrm{X}(a, b)$ with a cocycle in $H^1(\mathrm{X}; \mathbb{Z}^2)$.

We first build a system of generators for the homology of $\mathrm{X}(a, b)$. We label each copy of the torus fundamental domain in $\mathrm{X}(a, b)$ by 00, 01, 10 and 11 (see Figure 3a and 4). For $\kappa \in \{00, 10, 01, 11\}$ let h_κ (resp. v_κ) be the horizontal (resp. vertical) simple closed curve that delimit each copy (the exterior boundary). The curves h_κ (resp. v_κ) have holonomy 1 (resp. i). The automorphism group $K \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ of $\mathrm{X}(a, b)$ acts on the indices of h_κ and v_κ by addition (where we consider 0 and 1 as elements of $\mathbb{Z}/2$). The intersection form $\langle ., . \rangle$ on $\mathrm{X}(a, b)$ is such that $\langle h_\kappa, v_{\kappa'} \rangle = \delta_{\kappa,\kappa'}$ where δ_{ij} is the Kronecker symbol. In other words, the module generated by the elements h_κ and v_κ is a symplectic submodule and $\{(h_\kappa, v_\kappa)\}_\kappa$ is a symplectic basis. Moreover, this $\mathbb{Z}$-submodule is invariant under the action of K (but not irreducible, see Lemma 4 below).

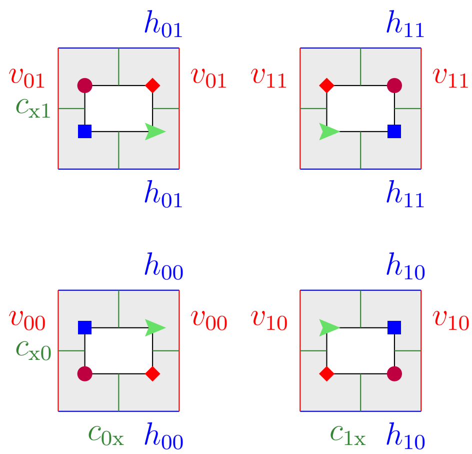


Figure 4: Homology generators for $\mathrm{X}(a, b)$.

We consider four more elements of $H_1(\mathrm{X}; \mathbb{Z})$. Let $c_{\mathrm{x}0}$ (resp. $c_{\mathrm{x}1}$) be the circumferences of the horizontal cylinder that intersects the two copies 00 and 10 (resp. 01 and 11) of the torus fundamental domain. The curves $c_{\mathrm{x}0}$ and $c_{\mathrm{x}1}$ have both holonomy $(2 - 2a, 0)$. We define as well the curves $c_{0\mathrm{x}}$ and $c_{1\mathrm{x}}$ with respect to the vertical cylinders. The curves

$c_{0\mathrm{x}}$ and $c_{1\mathrm{x}}$ have both holonomy $(0, 2-2b)$. As before the action of K as automorphism group of $\mathrm{X}(a,b)$ corresponds to an action on indices of c_{ij} if we set $0 \cdot \mathrm{x} = 1 \cdot \mathrm{x} = \mathrm{x}$.

There are two relations in $H_1(\mathrm{X};\mathbb{Z})$ among the curves defined above.

$$\begin{array}{l} c_{\mathrm{x}0} - c_{\mathrm{x}1} = h_{00} - h_{01} + h_{10} - h_{11} \\ c_{0\mathrm{x}} - c_{1\mathrm{x}} = v_{00} - v_{10} + v_{01} - v_{11} \end{array} . \tag{1}$$

We have the following elementary

Lemma 2. *The relations (1) are the only relations in the family $\{h_{ij}, v_{ij}, c_{\mathrm{x}j}, c_{i\mathrm{x}}\}$. Let E_h (resp. E_v) be the span of $\{h_{ij}, c_{\mathrm{x}k}\}_{i,j,k\in\{0,1\}}$ in $H_1(\mathrm{X}(a,b);\mathbb{Z})$ (res. of $\{v_{ij}, c_{k\mathrm{x}}\}_{i,j,k\in\{0,1\}}$ in $H_1(\mathrm{X}(a,b);\mathbb{Z})$) then $H_1(\mathrm{X}(a,b);\mathbb{Z}) = E_h \oplus E_v$ and the sum is orthogonal with respect to the intersection form.*

The infinite cover $\mathrm{X}_\infty(a,b) \to \mathrm{X}(a,b)$ corresponds to a certain subgroup H of $\pi_1(\mathrm{X}(a,b))$ such that $\pi_1(\mathrm{X}(a,b))/H \simeq \mathbb{Z}^2$. But as the cover is normal and $\mathrm{Deck}(\mathrm{X}_\infty(a,b)/\mathrm{X}(a,b)) \simeq \mathbb{Z}^2$ is an Abelian group, there exists a factorisation through the Abelianisation $H_1(\mathrm{X}(a,b);\mathbb{Z})$ of $\pi_1(\mathrm{X}(a,b))$ (see also [HoWe] for the description of $\mathbb{Z}$-cover). In other terms the cover is defined by an element of $H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ and more precisely we have the following explicit description.

Lemma 3. *The $\mathbb{Z}^2$ covering $\mathrm{T}(a,b)/\mathrm{X}(a,b)$ is given by the dual with respect to the intersection form of the cycle*

$$f = \begin{pmatrix} v_{00} - v_{10} + v_{01} - v_{11} \\ h_{00} - h_{01} + h_{10} - h_{11} \end{pmatrix} \in H_1(X;\mathbb{Z}^2).$$

In other words, the subgroup of $\pi_1(\mathrm{X}(a,b))$ associated to the covering is the kernel of

$$\pi_1(\mathrm{X}(a,b)) \xrightarrow{Ab} H_1(\mathrm{X}(a,b);\mathbb{Z}) \xrightarrow{\langle f,.\rangle} \mathbb{Z}^2.$$

Proof. As before we consider the surface decomposed into four copies of the torus fundamental domain labelled 00, 10, 01 and 11. Let γ be a smooth curve in $\mathrm{T}(a,b)$ which follows the law of reflection when it hits an obstacle. Let $\overline{\gamma}$ its image in $\mathrm{X}(a,b)$. There is an ambiguity for the starting point of $\overline{\gamma}$ and we assume that we start in the copy 00. Each time the curve $\overline{\gamma}$ hit a side associated to a vertical (resp. horizontal) scatter the curve $\overline{\gamma}$ switches from the copy κ to $(1,0)\cdot\kappa$ (resp. $(0,1)\cdot\kappa$). At the same time, in the infinite table $\mathrm{T}(a,b)$ the curve γ is reflected vertically (resp. horizontally). When the curve crosses a vertical (resp. an horizontal) boundary of the fundamental domain (labelled A (resp. B) in Figure 2a) the curve $\overline{\gamma}$ remains in the same copy. In other words, the endpoint of γ in $T(a,b)$ only depends on the monodromy of $\overline{\gamma}$ with respect to $\mathrm{X}_\infty/\mathrm{X}$ and we need to consider only the case of the curves $\gamma = h_{ij}, v_{ij}$ for $i = 0,1$ and $j = 0,1$.

As the copies 00 and 01 in $\mathrm{X}(a,b)$ corresponds to the absence of vertical reflection, the monodromy of v_{00} and v_{01} is $(1,0)$. Whereas for the copies 10 and 11, the curve γ has been reflected and the monodromy of h_{10} and h_{11} is $(-1,0)$. The same analysis can be made for the curves v_{ij} and the lemma follows from duality between $\{h_\kappa\}$ and $\{v_\kappa\}$. $\square$

Now, we use the description of $\mathrm{X}_\infty(a,b) \to \mathrm{X}(a,b)$ in terms of homology to approximate the distance $d(p, \phi_t^\theta(p))$ of Theorem 1 in terms of intersection. But first of all, we need to approximate geodesic segment by elements of $H_1(\mathrm{X},\mathbb{Z})$.

For each triple $(p, \theta, t) \in \mathrm{X} \times S^1 \times \mathbb{R}_+$ we define an element $\gamma_t^\theta(p) \in H_1(\mathrm{X}; \mathbb{Z})$ as follows. Consider the geodesic segment of length t from p in the direction θ and close it by a small piece of curve that does not intersect any curves h_κ nor v_κ. The curve used to close the geodesic segment can be chosen to be uniformly bounded.

The proposition below shows that the distance of the particle in the billiard $\mathrm{T}(a, b)$ can be reduced to the study of the intersection of the approximative geodesic $\gamma_t^\theta(p)$ in $\mathrm{X}(a, b)$.

Proposition 1. *Let $\|.\|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $p \in \mathrm{X}(a, b)$ be a point in the copy $(0, 0)$ of the fundamental domain, $\tilde{p} \in T(a, b)$ the lift of p which belongs to the copy $(0, 0)$ of translate of the fundamental domain and f as in the previous lemma. Then*

$$\left\| \left\langle f, \gamma_t^\theta(p) \right\rangle - \phi_t^\theta(\tilde{p}) \right\|_2 \leq \sqrt{2}.$$

In particular

$$\left| \|\langle f, \gamma_t^\theta(p) \rangle\|_2 - d(\tilde{p}, \phi_t^\theta(\tilde{p})) \right| \leq \sqrt{2}.$$

Proof. The distance between the point $\phi_t^\theta(p) \in \mathbb{R}^2$ and the associated level $\langle f, \gamma_t^\theta(p) \rangle \in \mathbb{Z}^2$ is bounded from above by the diameter of the fundamental domain. The latter is uniformly bounded by $\sqrt{2}$ (with respect to the parameters a and b). $\square$

As a consequence of the above proposition we reformulate our main result (Theorem 1).

Theorem 6. *Let $0 < a < 1$, $0 < b < 1$ and $\gamma_T^\theta(p)$ be the approximative geodesic starting from p in direction θ in $\mathrm{X}(a, b)$.*

1. *If a and b are rational numbers or can be written as $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x) + y\sqrt{D}$ with $x, y \in \mathbb{Q}$ and $D > 1$ a positive square-free integer, then for almost every θ and every point p in $\mathrm{X}(a, b)$ (with an infinite forward orbit under the linear flow)*
$$\limsup_{T \to \infty} \frac{\log \left| \left\langle f, \gamma_T^\theta(p) \right\rangle \right|}{\log T} = \frac{2}{3}.$$
2. *For almost all $(a, b) \in (0, 1)^2$, for almost all θ and for every point p in $\mathrm{X}(a, b)$ (with an infinite forward orbit)*
$$\limsup_{T \to \infty} \frac{\log \left| \left\langle f, \gamma_T^\theta(p) \right\rangle \right|}{\log T} = \frac{2}{3}.$$

3.3 Deck group action on $\mathrm{X}(a, b)$

We study the covering $\mathrm{X}(a, b)/\mathrm{L}(a, b)$ which is normal with Deck group the Klein four group $K = \mathbb{Z}/2 \times \mathbb{Z}/2$ by Lemma 1.

Let v_{ij}, h_{ij}, $c_{\mathrm{x}j}$ and $c_{i\mathrm{x}}$ for $i, j \in \{0, 1\}$ be the generators of $H_1(\mathrm{X}; \mathbb{Z})$ defined in Section 3.2. The action of the Klein four group K on $\mathrm{X}(a, b)$ splits the homology in four subspaces. For the generators $\tau_v = (1, 0)$ and $\tau_h = (0, 1)$ of K we define the subspace E^{+-} to be the set of vectors $v \in H_1(\mathrm{X}; \mathbb{Z})$ such that $\tau_v(v) = +1$ and $\tau_v(v) = -1$. We define similarly E^{++}, E^{-+} and E^{--}.

We note $h_K = h_{00} + h_{01} + h_{10} + h_{11}$ and $v_K = v_{00} + v_{01} + v_{10} + v_{11}$.

Lemma 4. *The action of the deck group of* $\mathrm{X}(a,b) \to \mathrm{L}(a,b)$ *splits the cohomology into four subspaces*

$$H^1(\mathrm{X}(a,b);\mathbb{Q}) = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{++},$$

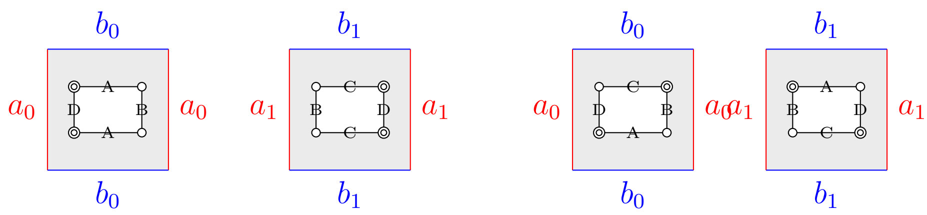
where each subspace E^κ *is defined over* $\mathbb{Q}$ *as follows*

- $E^{++} = \mathbb{Q}\,[h_K] \oplus \mathbb{Z}\,2\,[c_{\mathrm{x}0} + c_{\mathrm{x}1}] \oplus \mathbb{Q}\,[v_K] \oplus \mathbb{Q}\,2\,[c_{0\mathrm{x}} + c_{1\mathrm{x}}] \simeq H_1(\mathrm{L}(a,b);\mathbb{Q})$
- $E^{+-} = \mathbb{Q}\,[h_{00} - h_{01} + h_{10} - h_{11}] \oplus \mathbb{Q}\,[v_{00} - v_{01} + v_{10} - v_{11}]$
- $E^{-+} = \mathbb{Q}\,[h_{00} + h_{01} - h_{10} - h_{11}] \oplus \mathbb{Q}\,[v_{00} + v_{01} - v_{10} - v_{11}]$
- $E^{--} = \mathbb{Q}\,[h_{00} - h_{01} - h_{10} + h_{11}] \oplus \mathbb{Q}\,[v_{00} - v_{01} - v_{10} + v_{11}]$

We emphasise that the invariant part of $H_1(\mathrm{X}(a,b);\mathbb{Z})$ under the subgroup $\langle \tau_v \rangle \subset K$ can be identified with $H_1(\mathrm{X}(a,b)/\langle \tau_v \rangle;\mathbb{Z})$. This is the main reason for which we consider each quotient of $\mathrm{X}(a,b)$ by the three subgroups of order two generated by τ_v, τ_h and $\tau_h\,\tau_v$.

Lemma 5. *The surfaces* $\mathrm{X}(a,b)/\langle \tau_v \rangle$ *and* $\mathrm{X}(a,b)/\langle \tau_h \rangle$ *belongs to the hyperelliptic component* $\mathcal{H}^{hyp}(2,2)$ *while the surface* $\mathrm{X}(a,b)/\langle \tau_v \tau_h \rangle$ *belongs to the hyperelliptic locus* $\mathcal{L} \subset \mathcal{H}^{odd}(2,2)$.

Proof. We see on the two figures below that the central symmetry in each polygonal representation of the surfaces $\mathrm{X}(a,b)/\langle \tau_v \rangle$ and $\mathrm{X}(a,b)/\langle \tau_v \tau_h \rangle$ gives rise to a non orientable linear involution on $\mathrm{X}(a,b)$. In both cases the quotient is a sphere. In the first one, the singularities are exchanged and hence $\mathrm{X}(a,b)/\langle \tau_v \rangle$ belongs to $\mathcal{H}^{hyp}(2,2)$ which corresponds to the orientation cover of quadratic differentials in $\mathcal{Q}(4,-1^8)$. While for $\mathrm{X}(a,b)/\langle \tau_v \tau_h \rangle$ the zeros are Weierstrass points and the surface belongs to the hyperelliptic locus $\mathcal{L} \subset \mathcal{H}^{odd}(2,2)$ which corresponds to the orientation cover of quadratic differentials in $\mathcal{Q}(1^2,-1^6)$. □



(a) Quotient of $\mathrm{X}(a,b)$ by τ_h.

(b) Quotient of $\mathrm{X}(a,b)$ by $\tau_h\tau_v$.

Figure 5: The quotients of degree 2 of $\mathrm{X}(a,b)$.

4 Moduli space and Lyapunov exponents

In this section, using McMullen classification of $\mathrm{SL}(2,\mathbb{R})$-invariant closed set and probability measures in $\mathcal{H}(2)$, we classify the possible closure $\overline{\mathrm{SL}(2,\mathbb{R})\cdot \mathrm{X}(a,b)}$ of $\mathrm{SL}(2,\mathbb{R})$-orbits of the surfaces $\mathrm{X}(a,b)$ in $\mathcal{H}(2^4)$. Each closure carries a unique $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measure and we compute the Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to it.

4.1 Moduli space and $\mathrm{X}(a,b)$

We recall that $\mathrm{X}(a,b) \in \mathcal{H}(2^4)$ is a cover of $\mathrm{L}(a,b) \in \mathcal{H}(2)$ (Lemma 1). This property is preserved by the action of $\mathrm{SL}(2,\mathbb{R})$ and more precisely the action of $\mathrm{SL}(2,\mathbb{R})$ is equivariant: for any $g \in \mathrm{SL}(2,\mathbb{R})$ the surface $g\cdot\mathrm{X}(a,b)$ is a cover of $g\cdot\mathrm{L}(a,b)$. Hence, all $\mathrm{SL}(2,\mathbb{R})$-orbits of $\mathrm{X}(a,b)$ belongs to the sublocus of $\mathcal{H}(2^4)$ which corresponds to particular covering of surfaces in $\mathcal{H}(2)$. This locus, which we denote by $\mathcal{G}$, is a closed $\mathrm{SL}(2,\mathbb{R})$-invariant subvariety of $\mathcal{H}(2^4)$ which is a finite cover of $\mathcal{H}(2)$. In particular, McMullen's classification Theorem for $\mathrm{SL}(2,\mathbb{R})$-invariant closed subset and probability measures (Theorem 4) holds for closure of $\mathrm{SL}(2,\mathbb{R})$-orbits of $\mathrm{X}(a,b)$.

The action of the Klein four-group K on surfaces $\mathrm{X}(a,b)$ and the splitting of Lemma 4 holds for any surface Y in $\mathcal{G}$. For any $Y\in\mathcal{G}$ we denote by $\overline{Y} = Y/K$ its quotient in $\mathcal{H}(2)$. We have maps $H_1(Y;\mathbb{R}) \to H_1(\overline{Y};\mathbb{R})$ (resp. $H^1(\overline{Y};\mathbb{R}) \to H^1(Y;\mathbb{R})$) which are equivariant with respect to the Kontsevich-Zorich cocycle. In particuliar the explicit decomposition in the first part of Lemma 4 remains valid for any surface Y in $\mathcal{G}$ as it depends only of the action of K. In particular, we get an $\mathrm{SL}_2(\mathbb{R})$-equivariant splitting of the Hodge bundle. But, as $H_1(\mathrm{X}(a,b);\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$ can only be identified locally, the explicit basis of homology we have exhibited for $\mathrm{X}(a,b)$ has no meaning for Y.

4.2 Computation of Lyapunov exponents

In this section we compute the individual Lyapunov exponents of the KZ cocycle for all $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic measures on $\mathcal{G}$. We denote by $E\to\mathcal{G}$ the Hodge bundle over $\mathcal{H}(2^4)$ restricted to $\mathcal{G}$.

Recall, that the KZ cocycle is symplectic. Hence, the Lyapunov exponents come by pair of opposites $(\nu,-\nu)$. In the following we call *non negative spectrum* of the KZ cocycle the non-negative numbers $1=\nu_1 > \nu_2 \geq \ldots \geq \nu_g$ such that the multiset $(\nu_1,\nu_2,\ldots,\nu_g,-\nu_g,\ldots,-\nu_1)$ are the Lyapunov exponents of the KZ cocycle. In our case, for any surface Y in $\mathcal{G}$ the Oseledets decomposition of $H_1(Y;\mathbb{R})$ respect the splitting $E = E^{++}\oplus E^{+-}\oplus E^{-+}\oplus E^{--}$. Moreover, the maximal Lyapunov exponent of the KZ cocycle, which equals 1, belongs to E^{++}. Hence the non negative spectrum can be written $\{1,\nu^{++},\nu^{+-},\nu^{-+},\nu^{--}\}$ where $\{1,\nu^{++}\}$ (resp. $\{\nu^{+-}\}$, $\{\nu^{-+}\}$ and $\{\nu^{--}\}$) is the non negative Lyapunov spectrum of the KZ cocycle restricted to E^{++} (resp. E^{+-}, E^{-+} and E^{--}).

Theorem 7. *For any* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure on* $\mathcal{G}$*:*

$$\nu^{++}=\nu^{--}=1/3 \qquad \textit{and} \qquad \nu^{+-}=\nu^{-+}=2/3.$$

Proof. We first consider the case of the rank 4 subbundle E^{++} which corresponds to invariant vectors under the action of the Klein four group K. E^{++} identifies with the pullback of the Hodge bundle over $\mathcal{H}(2)$ and in particular, we deduce from results of M. Bainbridge [Ba] (see also Theorem 5) that $\nu^{++} = 1/3$.

We now consider the case of the rank 2 subbundles E^{κ} for $\kappa \in \{--, +-, -+\}$. Lemma 5 implies that the subbundle $E^{++} \oplus E^{--}$ (resp. $E^{++} \oplus E^{+-}$ and $E^{++} \oplus E^{-+}$) can be identified to pullback of bundles over different covering loci in $\mathcal{H}^{hyp}(2,2)$ and $\mathcal{L} \subset H^{odd}(2,2)$. More precisely, the quotient map $Y \mapsto Y/\langle \tau_h \tau_v \rangle$ (resp. $Y \mapsto Y/\langle \tau_v \rangle$ and $Y \mapsto Y/\langle \tau_h \rangle$) induce an isomorphism between $E^{++} \oplus E^{--}$ (resp $E^{++} \oplus E^{+-}$ and $E^{++} \oplus E^{-+}$) and respectively $H_1(Y/\langle \tau_h \tau_v \rangle; \mathbb{Z})$ (resp. $H_1(Y/\langle \tau_h \rangle; \mathbb{Z})$ and $H_1(Y/\langle \tau_v \rangle; \mathbb{Z})$).

To compute the remaining Lyapunov exponents, we use twice Theorem 5. The loci $\mathcal{H}^{hyp}(2,2)$ and $\mathcal{L}$ comes from orientation coverings of surfaces in the quadratic strata respectively $\mathcal{Q}(4, -1^8)$ and $\mathcal{Q}(1^2, -1^6)$. For those two components we get that the sum of positive Lyapunov exponents are respectively

$$\nu_1 + \nu_2 + \nu_3 = \frac{2/3 + 6}{4} = 5/3 \qquad \text{for } \mathcal{H}^{hyp}(2,2)$$
$$\nu_1 + \nu_2 + \nu_3 = \frac{8}{4} = 2 \qquad \text{for } \mathcal{L} \subset \mathcal{H}^{odd}(2,2).$$

By subtracting $4/3 = 1 + 1/3$ to each of them that corresponds to the contribution of E^{++} we get that that $\nu^{--} = 1/3$ and $\nu^{+-} = \nu^{-+} = 2/3$. $\square$

5 Deviations for translation surfaces

In this section, we prove Theorem 2 which concerns growth of geodesics.

We recall notation from the introduction. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differential and μ a g_t-invariant ergodic measure on $\mathcal{H}(\alpha)$. We denote by $1 = \nu_1 > \nu_2 > \ldots > \nu_k$ the positive Lyapunov exponents of the KZ cocycle and for $X \in \mathcal{H}(\alpha)$ which is Oseledets generic

$$H^1(X;\mathbb{R}) = F_1^u \supset F_2^u \supset \ldots \supset F_k^u \supset F_{k+1}^u = F^c \supset F_k^s \supset \ldots \supset F_1^s \supset F_0^s = \{0\} \qquad (2)$$

the associated Oseledets flag. By Oseledets theorem, the decomposition (2) is measurable and is invariant under the Teichmüller flow.

We want to prove the following statement: for μ-almost all X which are Oseledets generic, for all $p \in X$ with infinite forward orbit and any norm on $H^1(X;\mathbb{R})$

1. for all $f \in F_i^u \backslash F_{i+1}^u$
$$\limsup_{T\to\infty} \frac{\log|\langle f, \gamma_T(p))\rangle|}{\log T} = \nu_i,$$
2. for $f \in F^c$
$$\limsup_{T\to\infty} \frac{\log|\langle f, \gamma_T(p)\rangle|}{\log T} = 0,$$
3. there exists a constant C, such that for $f \in F_i^s \backslash F_{i-1}^s$
$$\forall T \geq 0,\ |\langle f, \gamma_T(p)\rangle| \leq C.$$

We first notice that to prove Theorem 2, by ergodicity of the Teichmüller flow, it is enough to prove it for surfaces X belonging in a small open set of $\mathcal{H}(\alpha)$ of positive measure. The strategy is as follows. We build a small open set in which we have uniform estimates for the linear flows. Next, for a surface in this small open set we consider long pieces of trajectory under the linear flow that we decompose using the KZ cocycle. Then, using the uniform estimates, we get the lower and upper bounds.

5.1 Transversals for the Teichmüller flow

In order to code geodesics in individual surface we use Veech's construction of zippered rectangles [Ve1]. This construction is not defined directly on $\mathcal{H}(\alpha)$. Let $\mathcal{H}^{lab}(\kappa)$ be the set of equivalence classes of surfaces with one marked outgoing separatrix and enough points marked in order to forbid symmetry of individual surface. As $\mathcal{H}^{lab}(\alpha)$ is a finite cover of $\mathcal{H}(\alpha)$ dynamical properties of the Teichmüller flow and the Konstevich-Zorich cocycles does not change. The choice of $\mathcal{H}^{lab}(\alpha)$ ensure that

– we can follow points of individual surfaces,
– for $X \in \mathcal{H}^{lab}(\alpha)$ if Y is near X then there is a small continuous deformation of X which gives Y and the outgoing marked separatrix of X is identified to the one of Y.

The measure μ on $\mathcal{H}^{lab}(\alpha)$ is renormalised to remain a probability measure. In what follows $\mathcal{H}(\alpha)$ denotes $\mathcal{H}^{lab}(\alpha)$.

A surface in $\mathcal{H}(\alpha)$ is called *regular* if there is no saddle connection in both horizontal and vertical directions. In a regular surface the linear flow in vertical direction is minimal

(Keane's Theorem [Ke]). If there is a connection in vertical (resp. horizontal) direction then the forward (resp. backward) orbit for the Teichmüller flow goes to infinity. In particular, using Poincare recurrence theorem, we get that the set of regular surfaces is a set of full measure for μ.

Let X be a regular surface in the support of μ and $\Sigma \subset X$ the finite set of singularities of X. Following [Ve1], we decompose the surface into zippered rectangles. Recall that there is a marked outgoing separatrix in X. We consider the initial segment of length 1 on this separatrix that we identify with $[0,1]$. The Poincare map of the linear flow in this segment is an interval exchange transformation. There exists a canonical segment $I \subset X$ built from Rauzy induction([Ve1, Proposition 9.1]). The rectangles above each domain of continuity of the interval exchange transformation on I give a decomposition $X = \bigcup R_j$ where R_j are geodesic rectangles with horizontal sides inside I and vertical sides which contain singularities. The number of rectangles is $d = 2g - 2 + s - 1$ where g is the genus of X and s the number of singularities.

Let X be a regular surface in the support of μ and $X = \bigcup R_j$ its decomposition into zippered rectangle. The parameters of the zippered rectangles (lengths and heights of the rectangles) give local coordinates for $\mathcal{H}(\alpha)$ in a neighbourhood of X. Let $U \subset \mathcal{H}(\kappa)$ be an open set which contains X and for which the zippered rectangles obtained from X gives a chart of $\mathcal{H}(\alpha)$. In U, we have a trivialisation of the Hodge bundle and we identify all fibers with $H^1(X;\mathbb{R})$.

To each rectangle R_j on a surface Y in U is associated a curve $\zeta_j \subset Y \backslash \Sigma$ (up to homotopy in $Y \backslash \Sigma$) which corresponds to the Poincare map on the canonic interval of Y. The vertical holonomy of ζ_j is the height of R_j. The following is a classical fact.

Lemma 6. *The set $\{\zeta_j\}_{j=1,\dots,d}$ forms a basis of $H_1(Y \backslash \Sigma; \mathbb{Z})$.*

Let $Y \in U$ and $I \subset Y$ be the canonical transversal for the linear flow of Y. To a point p in I, we associate the sequence of return times $T_n = T_n(p)$ of the linear flow into I. Each curve $\gamma_{T_n}(p)$ have both ends in I and we close it using a small piece of the horizontal segment contained in I. For any $p \in I$ with infinite orbit and any n we have a unique decomposition as concatenation of curves

$$\gamma_{T_n}(p) = \zeta_{j_1}(p)\ \zeta_{j_2}(p)\ \zeta_{j_3}(p)\ \dots\ \zeta_{j_n}(p)$$

and hence

$$\gamma_{T_n}(p) = \sum_{j=1}^{d} m_{T_n,j}(p)\zeta_j \in H_1(S \backslash \Sigma).$$

Let $p \in Y$ with infinite backward and forward orbit. There is a unique point $p' \in I$ such that the orbit of p' under the linear flow goes to p before returning in I. For $T \geq 0$, we denote by $\gamma_T(p)$ the curve $\gamma_{T_n}(p')$ where $T_{n-1}(p') < T \leq T_n(p')$.

Lemma 7. *We can choose U in such way that there exist constants K_1 and K_2 such that for all $Y \in U$*

1. *for $j = 1, \dots, d$, the length l_j and height h_j of the rectangle $R_j = R_j(Y)$ satisfy $K_1^{-1} < h_j < K_1$ and $K_1^{-1} < l_j < K_1$,*
2. *for every point $p \in I$, the decomposition of the geodesic $\gamma_{K_2}(p) = \sum m_j \zeta_j \in H_1(Y \backslash \Sigma; \mathbb{Z})$ is such that no m_j is zero. In other words, any geodesic longer than K_2 goes through all rectangles R_j.*

Proof. We consider the interval exchange transformation on the segment I associated to Y. We recall that by doubling all points which are preimages of a discontinuity of the interval exchange gives a Cantor set $\widetilde{I}$. The interval exchange transformation is well defined on this Cantor set and is a homeomorphism which is semi-conjugated to the initial transformation on I [Ke].

To fulfill the first condition, it is enough to choose a relatively compact set U inside the chart given by rectangles. We prove that it is possible to satisfy the second one. Because of regularity, the linear flow of X is minimal (Keane's Theorem [Ke]). Let $I \subset Y$ be the segment associated to Y. For any $p \in I$ with infinite future orbit, there exists a time $T = T(p)$ such that the curve $\gamma_T(p)$ has visited all rectangles. We choose $T(p)$ to be the first return time of p in I with this property. The map $p \mapsto T(p)$ is locally constant on the Cantor set associated to the interval exchange transformation on I and uniformly bounded because of minimality. Hence, on Y, any curve of length longer than $K = \max_{p\in Y} T(p) < \infty$ goes through all rectangles. In a small neighborhood of Y, the rectangles associated to the time K are still rectangles and their heights have been modified continuously with respect to the surface. By choosing U small enough we may ensure that all rectangles of length less than K in Y are still rectangles in $Y' \in U$ and their heights are uniformly bounded by $K_2 = K + \varepsilon$ with $\varepsilon > 0$. □

By taking smaller U if necessary, we assume that it is "flow box" that contains X. Namely, U is identified with a transversal P to the Teichmüller flow containing Y times an interval $]-\varepsilon;\varepsilon[$. For $Y \in P$, we consider the *backward* return times $t_n = t_n(Y)$ of the surface Y in P. By ergodicity of μ, this is well defined for μ-almost all Y in P. We set $\zeta_j^{(n)} = \left(B^{(t_n)}\right)^* (g_{t_n})(\zeta_j)$. The segment I in $g_{t_n}Y$ becomes a segment of length $e^{-t_n}\|I\|$ in Y. The curve $\zeta_j^{(n)}$ corresponds to a long piece of geodesic $\gamma_T(p)$ which starts and ends in $I^{(n)}$.

5.2 Upper bound

Let X be a regular surface and P a transversal to the Teichmüller flow containing X. Let $Y \in P$ a surface which is recurent for the Teichmüller flow and Oseledets generic. Then, from Oseledets theorem we know that the intersection $\left\langle f, \zeta_j^{(n)} \right\rangle = \left\langle B^{(t_n)} f, \zeta_j \right\rangle$ is bounded by $\|B^{(t_n)} f\|$ times a constant. To prove that the bound still holds for a generic geodesic, we use the following lemma which decomposes any geodesic into small pieces of the form $\zeta_j^{(n)}$.

Lemma 8 ([Fo] Lemma 9.4, [Zo1] Proposition 8)**.** *Let X be a regular surface and P a transversal containing X as in the previous Section. Let $Y \in P$ be recurent for the Teichmüller flow and $\zeta_j^{(n)}$ be as above. Let $p \in I \subset Y$ be a point with infinite future orbit. For each $T \geq 0$ there exists an integer $n = n(T)$ and a decomposition*

$$\gamma_T(p) = \sum_{k=0}^{n} \sum_{j=1}^{d} m_j^{(k)} \zeta_j^{(k)} \qquad \text{in } H_1(Y;\mathbb{Z}),$$

which satisfies

1. *the $m_j^{(k)}$ are non negative integers for $k = 1, \ldots, n$ and $j = 1, \ldots, d$,*

2. $\sum_j m_j^{(n)} \neq 0$ *and* $K_1^{-1}\, e^{t_n} < T$ *,*

3. $K_1^{-1}\, e^{t_k} \leq \ell\left(\zeta_j^{(k)}\right) \leq K_1\, e^{t_k}$ *for* $j = 1, \ldots, d$*,*

4. $\sum_j m_j^{(k)} \leq 2(K_1)^2 \exp(t_{k+1} - t_k)$*.*

Proof. Let $I = I^{(0)} \subset Y$ be the segment of the interval exchange transformation associated to the zippered rectangles decomposition of Y. For $k \geq 1$, let $I^{(k)}$ be the subintervals of Y which are the image of $I(g_{t_k} Y) \subset g_{t_k}(Y)$ under g_{-t_k}.

We describe the so called *prefix-suffix* decomposition in symbolic dynamics for $\gamma_T(p)$. We assume that T is a return time of the linear flow in I and note $\gamma = \gamma_T(p)$. Let n be the largest k such that the closed curve γ crosses twice $I^{(k)}$. We may decompose $\gamma = r_-^{(n)}\, \gamma^{(n)}\, r_+^{(n)}$ where

- $r_-^{(n)}$ starts from p and ends in $I^{(n)}$,
- $\gamma^{(n)}$ is a non empty concatenation of $\gamma_j^{(n)}$,
- $r_+^{(n)}$ starts from $I^{(n)}$ and ends at p.

We choose $r_-^{(n)}$ and $r_+^{(n)}$ to be minimal and hence their length are smaller than $K_1 e^{t_n}$. On the other hand, by definition, each curve $\gamma_j^{(n)}$ is of length larger than $K_1^{-1} e^{t_n}$ and hence $T > K_1^{-1} e^{t_n}$.

We now proceed by induction and decompose $r_-^{(n)}$ and $r_+^{(n)}$ with respect to the other recurrence times $0 < t_k < t_n$. We assume that we built two curves $r_-^{(k)}$ and $r_+^{(k)}$ and two sequences $\gamma_-^{(k)}, \gamma_-^{(k+1)}, \ldots, \gamma_-^{(n-1)}$ and $\gamma_+^{(k)}, \gamma_+^{(k+1)}, \ldots, \gamma_+^{(n-1)}$ such that

$$\gamma = r_-^{(k)}\ \left(\gamma_-^{(k)} \gamma_-^{(k+1)} \ldots \gamma_-^{(n-1)}\right)\ \gamma^{(n)}\ \left(\gamma_+^{(n-1)} \ldots \gamma_+^{(k+1)} \gamma_+^{(k)}\right)\ r_+^{(k)}$$

with

- $\gamma_-^{(m)}$ starts from $I^{(m)}$, ends in $I^{(m+1)}$ and does not cross $I^{(m+1)}$ before its endpoint,
- $\gamma_+^{(m)}$ starts from $I^{(m+1)}$, ends in $I^{(m)}$ and does not cross $I^{(m+1)}$ after its startpoint,
- $r_-^{(k)}$ starts from $I^{(0)}$ and ends in $I^{(k)}$ and does not cross $I^{(k)}$ before its endpoint,
- $r_+^{(k)}$ starts from $I^{(k)}$ and ends in $I^{(0)}$ and does not cross $I^{(k)}$ after its startpoint.

We end with the decomposition

$$\gamma = \gamma_-^{(0)}\ \gamma_-^{(1)}\ \ldots\ \gamma_-^{(n-1)}\ \gamma^{(n)}\ \gamma_+^{(n-1)}\ \ldots\ \gamma_+^{(1)}\ \gamma_+^{(0)}.$$

From the construction, we know that both $\gamma_-^{(k)}$ and $\gamma_+^{(k)}$ decomposes on the basis $\zeta_j^{(k)}$. Let $\gamma_-^{(k)} + \gamma_+^{(k)} = \sum m_j^{(k)} \zeta_j^{(k)}$. By maximality $\gamma_-^{(k)}$ satisfy $\ell\left(\gamma_-^{(k)}\right) < K_1 e^{(k+1)}$ and the same is true for $\gamma_+^{(k)}$. On the other hand each $\zeta_j^{(k)}$ is of length at least $K_1^{-1} e^{t_k}$ and hence

$$K_1^{-1} e^{t_k} \sum m_j^{(k)} < \sum m_j^{(k)} \ell(\zeta_j^{(k)}) < 2 K_1 e^{t_{k+1}}.$$

The latter inequality implies that $\sum m_j^{(k)} < 2(K_1)^2 e^{t_{k+1} - t_k}$. □

Now, we prove the upper bound in Theorem 2. We restrict to the case 1 relative to one of the unstable subspace F_i^u of the Oseledets flag. The same proof works for the other cases. We follow mainly Section 9 of [Fo] (see also Section 6 of [Zo1] and Section 4.9 of [Zo2]). In what follows K_i for $i = 3, 4, \ldots$ denote constants which do not depend on the time T or the number $n = n(T)$.

We fix $Y \in P$ which is Oseledets generic and $p \in Y$ with infinite forward orbit for the linear flow. By Lemma 8, for any $\varepsilon > 0$, there exists a constant K_3 and such that for T big enough the following estimation holds

$$|\langle f, \gamma_T(p)\rangle| \leq K_3 \sum_{k=1}^{n} \exp(t_{k+1} - t_k)\ \exp((\nu_i + \varepsilon)t_k). \tag{3}$$

Moreover, if Y is Birkhoff generic for the Teichmüller flow, we have

$$\lim_{k\to\infty} \frac{t_k}{k} = M$$

where M is the inverse of the transverse μ-measure of P. We assume that Y is Birkhoff generic. For any δ the following estimation holds for k big enough

$$(M - \delta)\, k \leq t_k \leq (M + \delta)\, k \tag{4}$$

Using (3) and (4) we get that for T big enough we get

$$\begin{aligned}
|\langle f, \gamma_T(p)\rangle| &\leq K_4 \sum_{k=1}^{n} \exp\left((M+\delta)(k+1) - (M-\delta)k\right)\ \exp\left(\nu_i + \varepsilon)(M+\delta)k\right) \\
&\leq K_5 \sum_{k=1}^{n} \exp\left((\nu_i + \varepsilon)(M+\delta) + 2\delta)k\right) \\
&\leq K_6 \exp(((\nu_i + \varepsilon)(M+\delta) + 2\delta)n).
\end{aligned}$$

Now, by the choice of $n = n(T)$ in the estimate 2 in Lemma 8 we have for T big enough

$$\exp\left((M - \delta)n\right) \leq \exp(t_n) \leq K_1^{-1}\, T$$

Hence we get that for T big enough

$$\frac{\log |\langle f, \gamma_T(p)|}{\log T} \leq \frac{(\nu_i + \varepsilon)(M + \delta) + 2\delta}{M - \delta}.$$

As δ and ε can be chosen arbitrarily small we get the upper bound.

5.3 Lower bound

We now prove the lower bound in Theorem 2. The only non trivial case is the one of a cocycle in unstable part F_i^u or the central part F^c of the Oseledets flag (2). The proof is identical for both of them.

We first fix notations for the whole section. Let (X, ω) be a regular surface and P be a transversal containing X as in Section 5.1. Let $Y \in P$ be recurrent for the Teichmüller flow and Oseledets generic. We denote by t_n the sequence of return times in P. We fix a point $p \in Y$ with infinite future orbit and a cocycle $f \in F_i^u \backslash F_{i+1}^u$ in the unstable part of $H^1(X; \mathbb{Z})$.

We use a similar decomposition as in the proof of Lemma 8. For all n, we consider the return time of p into $I^{(n)}$. For the m-th return time T in $I^{(n)}$ we have a decomposition

$$\gamma_T(p) = \gamma_-^{(1)}\ \gamma_-^{(2)}\ \ldots\ \gamma_-^{(n-1)}\ \zeta_{j_1}^{(n)} \zeta_{j_2}^{(n)} \ldots \zeta_{j_m}^{(n)}.$$

where the sequence $(j_k) = (j_k(n))$ does only depends on p and n and the sequence $\gamma_-^{(k)}$ only on p. The length of the initial segment $\gamma_-^{(1)}\ \gamma_-^{(2)} \dots\ \gamma_-^{(n-1)}$ corresponds to the first hitting time T_n^- of $I^{(n)}$ starting from p.

Let $\varepsilon > 0$, we want to prove that the following holds

$$\limsup_{T\to\infty} \frac{\log |\langle f, \gamma_T(p)\rangle|}{\log T} \geq \nu_i - \varepsilon.$$

Either the above equality holds for the sequence of prefix $\gamma_-^{(1)}\ \dots\ \gamma_-^{(n-1)}$ and we are done. If not, for all n big enough

$$\log \left|\left\langle f, \gamma_{T_n^-}(p)\right\rangle\right| = \log \left|\left\langle f, \gamma_-^{(1)}\gamma_-^{(2)} \dots \gamma_-^{(n-1)}\right\rangle\right| \leq (\nu_i - \varepsilon/2) \log T_n^-. \tag{5}$$

Lemma 9. *There exists an index $\ell \in \{1, \dots, K_1K_2\}$, a constant $C > 0$ and an infinite subset $N \subset \mathbb{N}$ such that for all $n \in N$*

$$\forall \ell' \in \{1, \dots, \ell - 1\},\ \frac{\left|\left\langle B^{(t_n)} f, \zeta_{j_{\ell'}(n)}\right\rangle\right|}{\|B^{(t_n)} f\|} \leq \frac{C}{K_1K_2} \quad \textit{and} \quad \frac{\left|\left\langle B^{(t_n)} f, \zeta_{j_\ell(n)}\right\rangle\right|}{\|B^{(t_n)} f\|} \geq C.$$

Proof. From compactness of the sphere in $H^1(X; \mathbb{Z})$, there exists a constant $C' > 0$ depending only on the generating system $\{\zeta_j\}_{j=1,\dots,d}$ and the norm $\|.\|$ on $H^1(X; \mathbb{Z})$ such that

$$\forall v \in H^1(X; \mathbb{Z}), \quad \max_{j=1,\dots,d} |\langle v, \zeta_j\rangle| \geq C'\|v\|.$$

Now, we use the fact that before time K_2 all different curves ζ_j appear. It implies that before K_1K_2 return times in I any geodesic $\gamma_T(p)$ passes through all rectangles. In particular for at least one of the curves $\zeta_{j_1(n)}, \dots \zeta_{j_{K_1K_2}(n)}$ we have a uniform lower bound on the intersection with f.

Now, consider the sequence of pieces in first position $\zeta_{j_1(n)}$. If

$$\limsup_{n\to\infty} \frac{\left|\left\langle B^{(t_n)} f, \zeta_{j_1(n)}\right\rangle\right|}{\|B^{(t_n)} f\|} > 0$$

then we are done by choosing C to be the half of the lim sup above. If not, we consider the sequence of pieces in second position $\zeta_{j_2(n)}$ and repeat the dichotomy. We know from the first part of the proof, that this process stops before the (K_1K_2)-th position. We get a position ℓ, a constant C, and a subsequence $N \subset \mathbb{N}$ that satisfy the right inequality of the statement of the lemma. By starting the subsequence far enough (i.e. considering $N \cap \{m, m+1, \dots\}$ for m big enough), we may ensure by our construction that the $\ell - 1$ inequalities on the left holds. □

Let ℓ, C and $N \subset \mathbb{N}$ that satisfies the conclusion of Lemma 9. Let $n \in N$ and $p_n \in I^{(n)}$ be the endpoint of the prefix $\gamma_-^{(1)}\ \dots\ \gamma_-^{(n-1)}$. Then

$$\begin{aligned}
\langle f, \gamma_{T_n}(p)\rangle &= \left\langle f, \gamma_{T_n^-}(p)\right\rangle + \left\langle f, \gamma_{T_n - T_n^-}(p_n)\right\rangle \\
&= \left\langle f, \gamma_-^{(1)}\ \dots\ \gamma_-^{(n-1)}\right\rangle + \left\langle f, \zeta_{j_1(n)}^{(n)}\right\rangle + \dots + \left\langle f, \zeta_{j_{l-1}(n)}^{(n)}\right\rangle + \left\langle f, \zeta_{j_l(n)}^{(n)}\right\rangle \\
&= \left\langle f, \gamma_-^{(1)}\ \dots\ \gamma_-^{(n-1)}\right\rangle + \left\langle B^{(t_n)} f, \zeta_{j_1(n)}\right\rangle + \dots + \left\langle B^{(t_n)} f, \zeta_{j_{l-1}(n)}\right\rangle + \left\langle B^{(t_n)} f, \zeta_{j_l(n)}\right\rangle.
\end{aligned}$$

Using triangular inequality, we get

$$
\begin{aligned}
|\langle f, \gamma_{T_n}(p)\rangle| &\geq \left|\left\langle B^{(t_n)} f, \zeta_{j_l(n)}\right\rangle\right| - \sum_{k=1}^{l-1} \left|\left\langle B^{(t_n)} f, \zeta_{j_k(n)}\right\rangle\right| - \left|\left\langle f, \gamma_{T_n^-}(p)\right\rangle\right| \\
&\geq C \left\|B^{(t_n)} f\right\| - \frac{(l-1)C}{K_1 K_2} \left\|B^{(t_n)} f\right\| - \left|\left\langle f, \gamma_{T_n^-}(p)\right\rangle\right| \\
&\geq \frac{C}{K_1 K_2} \left\|B^{(t_n)} f\right\| - \left|\left\langle f, \gamma_{T_n^-}(p)\right\rangle\right|.
\end{aligned}
$$

We use twice Lemma 7 to prove that T_n growth like e^{t_n}. First of all, as T_n is a time for which the orbit of p under the linear flow has reached at least twice the interval $I^{(n)}$ we have $T_n > K_1^{-1} e^{t_n}$. And, by construction, $T_n < K_2 e^{t_n}$. Hence

$$
\lim_{n \to \infty} \frac{\log T_n}{t_n} = 1. \tag{6}
$$

From our assumption (5), for $n \in N$ big enough, the term $|\langle f, \gamma_{T_n^-}\rangle|$ is exponentially smaller than $\|B^{(t_n)} f\|$. We hence get that

$$
\limsup_{T \to \infty} \frac{|\langle f, \gamma_T(p)\rangle|}{\log T} \geq \limsup_{n \in N} \frac{\log \left(C/K_2 \left\|B^{(t_n)} f\right\| - \left|\left\langle f, \gamma_{T_n^-}(p)\right\rangle\right|\right)}{t_n} = \limsup_{n \in N} \frac{\log \left\|B^{(t_n)} f\right\|}{\log t_n} = \nu_i.
$$

In other words, under assumption (5) we exhibit a subsequence on which the lim sup is achieved.

5.4 Theorem 6 for the whole locus $\mathcal{G}$

We now prove the first step of our main theorem about the wind-tree model in the following form

Lemma 10. *Let $0 < a < 1$ and $0 < b < 1$. Let μ be the $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic measure on $\mathcal{G}$ which is supported on the adherence of the surface $\mathrm{X}(a,b)$. In a neighbourhood U of $\mathrm{X}(a,b)$ for which there is a trivialisation of the Hodge bundle. Let $f \in H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ be the cocycle that defines the wind-tree model. We consider f as a section of the Hodge bundle over U. Then for almost all $Y \in U$, every point $p \in Y$ with infinite forward orbit*

$$
\limsup_{T \to \infty} \frac{\log |\langle f, \gamma_T(p)\rangle|}{\log(T)} = \frac{2}{3}.
$$

Proof. From Lemma 4, we know that the cocycle $f \in H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ decomposes into two pieces $f^{+-} \in E^{+-}$ and $f^{-+} \in E^{-+}$ where each of E^{+-} and E^{-+} are rank 2 subbundles stable under the Kontsevich-Zorich cocycle. From Theorem 7, the Lyapunov exponents of the KZ cocycle in both of E^{+-} and E^{-+} are $2/3$ and $-2/3$. The only thing to prove in Lemma 10 is that f^{+-} (resp. f^{-+}) does not belong to the stable subspace of E^{+-} (resp. E^{-+}) associated to $-2/3$. If f belongs to the stable subspace, then $\|B^{(t)} f\|$ goes to zero as t goes to infinity. But recall that the the Kontsevich-Zorich cocycle takes values in the set of integer matrices of determinant 1 and that f is an integer cocycle. In particular, the quantity $\|B^{(t)} f\|$ is bounded below by

$$
\min_{v \in E^{+-} \setminus \{0\}} \|v\|.
$$

□

6 From deviations for generic surfaces to deviations for $\mathrm{X}(a,b)$

In this section we prove that Lemma 10 implies Theorem 6. We emphasise that in Lemma 10, we proved a statement on deviations which is valid for a generic surface in $\mathcal{G}$ but not necessarily for $\mathrm{X}(a,b)$. Using properties of the KZ cocycle we prove that the theorem holds for $\mathrm{X}(a,b)$.

Let $\mathrm{X}(a,b)$ be a surface and U a neighbourhood of $\mathrm{X}(a,b)$ in the closure of its $\mathrm{SL}(2,\mathbb{R})$ orbit. For a translation surface $(Y,\omega) \in U$ and $y \in Y$ we consider the function

$$F(\omega,p) = \limsup_{T\to\infty} \frac{\log|\langle f, \gamma_T(p), v\rangle|}{\log(T)}$$

We summarise what is already proven: the quantity $F(Y,p)$ is equal to $2/3$ for almost every $Y \in U$ and every $p \in Y$ with infinite orbit under the linear flow.

Remark 1. *To simplify notations, in this section, the sentence for every $p \in Y$ will mean for every $p \in Y$ with infinite forward orbit.*

Lemma 11. *The function F is invariant under the Teichmüller flow g_t and depends locally only on the cohomology class of* $[\mathrm{Re}(\omega)] \in H^1(Y;\mathbb{R})$.

Proof. The Teichmüller flow contracts time of the vertical flow but sends the vertical foliation to vertical and does not change the value of F. More generally, if the vertical foliation does not change (i.e. $[\mathrm{Re}(\omega)]$ does not change) then F remains constant. □

We first consider the case of the $\mathrm{SL}(2,\mathbb{R})$-invariant measure μ supported on a Teichmüller curve $\mathcal{C}$ in $\mathcal{G}$. In that case, a neighbourhood of $\mathrm{X}(a,b)$ in $\mathcal{C}$ is given by its $\mathrm{SL}(2,\mathbb{R})$ neighbourhood which is $\{h_s g_t r_\theta \mathrm{X}(a,b)\}$ for $(\theta,s,t) \in (-\varepsilon,\varepsilon)^3$.

Lemma 12. *For almost every θ, for every $x \in \mathrm{X}(a,b)$, $F(r_\theta \mathrm{X}(a,b),x) = 2/3$.*

Proof. By Lemma 11, the function F is invariant under the horocycle flow h_s which preserves $[\mathrm{Re}(\omega)]$.

Assume by contradiction that there is a set of positive measure $\Theta \subset S^1$, such that, for $\theta \in \Theta$ there is a point $p_\theta \in \mathrm{X}(a,b)$ with $F(r_\theta \mathrm{X}(a,b), p_\theta) \neq 2/3$.

The set $\Omega = \{h_s g_t r_\theta \in \mathrm{SL}(2,\mathbb{R}); (s,t) \in (-\varepsilon,\varepsilon)^2,\ \theta \in \Theta\}$ has positive measure in $\mathrm{SL}(2,\mathbb{R})$. By invariance of F under the geodesic flow and the horocycle flow, for every $g \in \Omega$, there is a point p_g in $g \cdot \mathrm{X}(a,b)$ such that $F(g \cdot \mathrm{X}(a,b), p_g) \neq 2/3$. This is a contradiction with the generic result for F. □

The case of the $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $\mathcal{G}$ obtained from the Lebesgue measure on $\mathcal{H}(2)$ is similar to the one for Teichmüller curves.

Lemma 13. *Let $\omega_{a,b,\theta}$ be the Abelian differential on the translation surface $r_\theta \cdot \mathrm{L}(a,b)$ then the map $(\theta,a,b) \mapsto [\mathrm{Re}(\omega_{\theta,a,b})] \in \mathbb{P}(H^1(\mathrm{L}(a,b);\mathbb{R}))$ is locally injective and open.*

Proof. The three parameters θ, a and b are clearly independent as it can be checked on an interval exchange transformations associated to a Poincare map of the linear flow. □

The proof then follows from the same argument as the one in the proof of Lemma 12 and ends the proof of the second case in Theorem 1.

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Annexe C

Cardinalité des classes de Rauzy

Dans cette deuxième annexe, nous reproduisons une version de l'article sur la cardinalité des classes de Rauzy. La numérotation des pages tout au long de l'article suit celle de l'article et non de la thèse. Ainsi, la page 111 de cette thèse est numérotée 1. Cette numérotation spéciale sera en italique.

Cardinality of Rauzy classes

Abstract

Rauzy classes form a partition of irreducible partitions. They were introduced as part of a renormalization algorithm for interval exchange transformations. We prove an explicit formula for the cardinality of each Rauzy class. Our proof uses a geometric interpretation of permutations and Rauzy classes in terms of translation surfaces and their moduli spaces.

Résumé

Cardinalité des classes de Rauzy

Les classes de Rauzy sont des sous-ensembles des permutations irréductibles qui forment une partition. Elles ont été introduites par G. Rauzy dans l'étude d'un algorithme de renormalisation des échanges d'intervalles. Nous démontrons une formule explicite pour la cardinalité de chaque classe de Rauzy. La preuve que nous développons utilise une interprétation géométrique des permutations et des classes de Rauzy en termes de surfaces de translation et de leurs espaces de modules.

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n$ be a vector with positive coordinates and $\pi \in S_n$ be a permutation. For $i = 2, \ldots, n+1$, we define $x_i = \sum_{j=1}^{i-1} \lambda_j$ and $y_i = \sum_{j=1}^{i-1} \lambda_{\pi^{-1}(j)}$ and note $x_1 = y_1 = 0$ and $a = x_{n+1} = y_{n+1}$. The interval exchange transformation $T = T_{\lambda,\pi}$ with data (λ, π) is the map defined on $[0,\ a)$ to itself, by

$$T(x) = x - x_i + y_{\pi(i)} \quad \text{if } x \in [x_i,\ x_{i+1}).$$

In other words, on the subinterval $[x_i,\ x_{i+1})$, the map T acts as a translation by $y_{\pi(i)} - x_i$. An interval exchange transformation is bijective and right continuous. The map T is an example of measurable dynamical system as it preserves the Lebesgue measure on $[0,\ a)$.

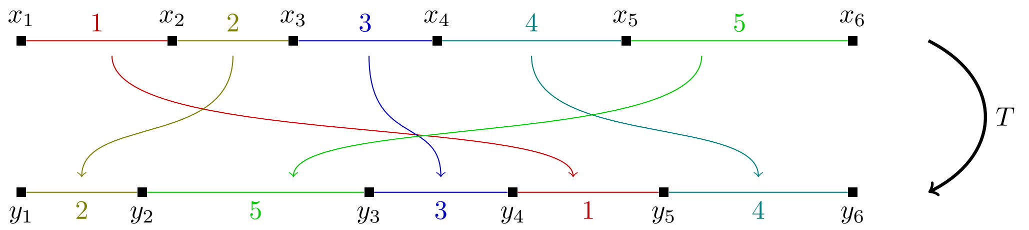

Figure 1.1: An interval exchange transformation with permutation $\pi = \binom{1\,2\,3\,4\,5}{2\,5\,3\,1\,4}$.

If $\pi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$ for k such that $1 \leq k < n$, then the two subintervals $[0,\ x_{k+1})$ and $[x_{k+1},\ a)$ are invariant under $T_{\lambda,\pi}$. We are interested in permutations that do not allow such a splitting.

Definition 1.1. A permutation $\pi \in S_n$ is *irreducible* (or *indecomposable*) if there is no k, $1 \leq k < n$, such that $\pi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$.

We denote by S_n^o the set of irreducible permutations in S_n. It was proved by M. Keane [Kea75] that if $\pi \in S_n^o$ then for Lebesgue almost all $\lambda \in \mathbb{R}_+^n$ the interval exchange $T_{\lambda,\pi}$ is minimal. Later H. Masur [Mas82] and W.A. Veech [Vee82], independently, proved that for Lebesgue almost all $\lambda \in \mathbb{R}_+^n$ the interval exchange $T_{\lambda,\pi}$ is uniquely ergodic.

In order to study the dynamics of interval exchange transformations, [Rau79] defines an induction procedure (named *Rauzy induction*) on the space of interval exchange transformations. In other words, a map $R : S_n^o \times \mathbb{R}_+^n \to S_n^o \times \mathbb{R}_+^n$. There are two cases of induction depending whether $x_n < y_n$ (*top induction*) or $x_n > y_n$ (*bottom induction*). The induction is not defined if $x_n = y_n$. Let $T_{\lambda,\pi}$ be an interval exchange transformation with $x_n \neq y_n$ and $T_{\lambda',\pi'} = R\left(T_{\lambda,\pi}\right)$ the one obtained by Rauzy induction. The permutation π' only depends on the type of the induction. Hence, there are two combinatorial operations $R_t : S_n^o \to S_n^o$ (top induction) and $R_b : S_n^o \to S_n^o$ (bottom induction) which corresponds to the operation on the permutation π associated to the Rauzy induction. The equivalence classes induced by the action of R_t and R_b on S_n^o are called *Rauzy classes*.

As far as we know, up today, only the Rauzy class $\mathcal{R}_n^{sym}$ of the symmetric permutation $\pi_n^{sym} \in S_n^o$ defined by $\pi_n^{sym}(k) = n - k + 1$ for $k = 1, \ldots, n$ has been described so far [Rau79]. In particular G. Rauzy proved that its cardinality is $|\mathcal{R}_n^{sym}| = 2^{n-1} - 1$. Computer experimentations have been made by P. Arnoux in the 80's, M. Kontsevich and A. Zorich in the 90's in relation to the classifcation of connected components of strata of the moduli space of Abelian differentials. Motivated by the study of [Rau79], the aim of this article is to study the combinatorics of Rauzy classes in S_n^o and establish a formula for their cardinalities.

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I wish to thank Corentin Boissy, Erwan Lanneau and Arnaldo Nogueira for their patient lectures and comments and Samuel Lelièvre for his help on drawing pictures with the PGF Ti*k*Z library for LaTeX. All formulas in the article have been tested with the help of the mathematical software Sage [Sa].

2 Main results

We recall elements from Teichmüller theory which yield to a classification of Rauzy diagrams. Let $I = [0, a)$ be an interval and T a map from I into itself. Let $f : [0, a) \to \mathbb{R}_+$ and X be the quotient of $\{(x, y) \in [0, a) \times \mathbb{R}_+;\ y \leq f(x)\}$ by the relation $(x, f(x)) \sim (T(x), 0)$. The space X together with the flow ϕ_t in the vertical direction is called a *suspension* and f the *roof function*. The flow ϕ_t has the property that the first return map on the interval $I \times \{0\} \subset X$ is exactly the map T. W. Veech [Vee82] considered roof functions which are constant on each subinterval of an interval exchange transformation. The suspension X obtained by this procedure is a translation surface and the flow ϕ_t corresponds to the geodesic flow on X in the vertical direction. Translation surfaces are part of Teichmüller theory and will play an important role in the construction of our counting formulas.

A translation surface S has a flat metric, except at a finite number of points where there are conical singularities whose angles are integer multiples of 2π. If S is a suspension of an interval exchange transformation T, the conical singularities of S come from the singularities of T. Let $n_1 2\pi, n_2 2\pi, \ldots, n_k 2\pi$ be the list of angles of the conical singularities of S. We call the integer partition $p = (n_1, n_2, \ldots, n_k)$ the *profile* of S. The genus g of S is related to p by $2g - 2 = \sum_{i=1}^{k} (n_i - 1) = s(p) - l(p)$ where $s(p) = n_1 + \ldots + n_k$ is the *sum* of p and $l(p) = k$ its *length*. We emphasize that the suspension associated to an interval exchange transformation is not unique but all of them have the same profile. Furthermore, suspensions obtained from permutations in the same Rauzy class have the same profile. Let $\Omega\mathcal{M}_g$ be the moduli space of translation surfaces of genus g and $p = (n_1, n_2, \ldots, n_k)$ an integer partition such that $2g - 2 = \sum_{i=1}^{k} (n_i - 1)$. The *stratum with profile* p denoted $\Omega\mathcal{M}_g(p)$ is the subset of $\Omega\mathcal{M}_g$ made out of surfaces whose profiles are p. On $\Omega\mathcal{M}_g$ acts the Teichmüller flow which preserves strata and for which the Rauzy-Veech induction on suspensions is viewed as a first return map. There is a bijection between extended Rauzy classes and connected components of strata $\Omega\mathcal{M}(p)$ [Vee82] where an *extended Rauzy class* is an equivalence class of irreducible permutations under the action of R_t, R_b and s, where s is the operation which acts on π in S_n by $s(\pi)(k) = \pi^{-1}(n - k - 1)$. W. Veech [Vee82, Vee90] and C. Boissy [Boi09] proved a bijection between Rauzy classes and connected components of strata with a choice of a part of the profile p. This choice corresponds to the marking of suspension induced by the left endpoint of the interval exchange transformation which is not affected during the Rauzy inductions. The combinatorial question of classifying (extended) Rauzy classes is hence translated into the geometric one of classifying connected components. M. Kontsevich and A. Zorich [KZ03] classified connected components of strata in terms of geometrical invariants: the *spin parity* (an element of $\{0, 1\}$) and the *hyperellipicity*. A spin parity occurs when the profile p has only odd parts and give rise to at least two distinct connected components. The term hyperellipticity stands for a serie of connected components that appear for the profiles $(2g-1, 1^k)$ and $(g, g, 1^k)$. This yields to a classification of (extended) Rauzy classes.

Our approach to count permutations in Rauzy classes relies in the above geometric interpretation of Rauzy classes. Let $\pi \in S_n^o$ be an irreducible permutation and p_π the profile of a suspension of π. The profile does not reflect the structure of an embedded segment in a surface and we refine the notion. We say that π has *marking* $m|a$ if the extremities of the interval corresponds to the same singularity P in the suspension which has a conical angle $m2\pi$ and $a \in \{0, \dots, m-1\}$ is such that $(2a+1)\pi$ is the angle between the left part and the right part of the interval measured from P. It has *marking* $m_l \odot m_r$ if the two extremities of the interval correspond to two different singularities of angles $m_l 2\pi$ on the left and $m_r 2\pi$ on the right. The data which consists of the profile and the marking is called the *marked profile* of the permutation π. We denote by $(m|a, p')$ (resp. $(m_l \odot m_r, p')$) a profile $p' \uplus (m)$ (resp. $p' \uplus (m_l, m_r)$) with marking $m|a$ (resp. $m_l \odot m_r$). Here $\uplus$ stands for the disjoint union of partitions considered as multisets. Our main theorem (see below) is a recurrence formula for the number of irreducible permutations with given marked profile.

We first consider standard permutations introduced in [Rau79].

Definition 2.1. A permutation $\pi \in S_n$ is *standard* if $\pi(1) = n$ and $\pi(n) = 1$.

A standard permutation is in particular irreducible. Those permutations were used for dynamical purpose in [NR97] and [AF07] in order to prove the weak mixing property of interval exchange transformations and in [KZ03],[Zor08] and [Lan08] in the study of connected components of strata. In terms of moduli space of translation surfaces, a standard permutation corresponds to a so called *Strebel differential*.

Let $\mathbf{p}$ be a marked profile whose profile is p. We denote by $\gamma^{std}(\mathbf{p})$ the number of standard permutations with marked profile $\mathbf{p}$. Moreover, if p has only odd terms, we define $\delta^{std}(\mathbf{p}) = \gamma_1^{std}(\mathbf{p}) - \gamma_0^{std}(\mathbf{p})$ where $\gamma_s^{std}(\mathbf{p})$ denotes the number of standard permutations with marked profile profile $\mathbf{p}$ and spin parity s. We prove explicit formulas for $\gamma^{std}(\mathbf{p})$ and $\delta^{std}(\mathbf{p})$. The formulas involve the numbers z_p, $c(p)$ and $d(p)$ which are defined next but we first introducte notations for partitions. Let $p = (n_1, n_2, \dots, n_k)$ be an integer partition considered as a multiset (each part has a multiplicity equals to its number of occurences in the partition). We recall that the disjoint union is denoted $\uplus$. We have $s(p_1 \uplus p_2) = s(p_1) + s(p_2)$ and $l(p_1 \uplus p_2) = l(p_1) + l(p_2)$. If $q \subset p$ is a subpartition of p we denote by $p \backslash q$ the unique partition r such that $p = q \uplus r$.

We recall that the conjugacy classes of S_n are in bijection with integer partitions of n. We denote by z_p the cardinality of the conjugacy class associated to p. If e_i is the number of occurences of i in p, then

$$z_p = \prod_{i=1}^{n} e_i! \; i^{e_i}.$$

If p satisfies $s(p) + l(p) \equiv 0 \mod 2$ we define (the formula is due to G. Boccara [Boc80])

$$c(p) = \frac{2(n-1)!}{n+1} \left(\sum_{q \subset (n_2, n_3, \dots, n_k)} (-1)^{s(q)+l(q)} \binom{n}{s(q)}^{-1} \right)$$

where the summation is over all subpartition of $(n_2, n_3, \dots, n_k)$ with multiplicity in the sense that $(1,3)$ occurs twice in $(1,1,3)$. Moreover, if the partition p has only odd parts we define $d(p) = (n-1)!/2^g$ where $g = (s(p) - l(p))/2$.

Our proofs are based on surgeries of partitions which are used to obtain recurrence (with a geometric counterpart as in [KZ03] and [EMZ03]). If m is a part of p and $a \in \{0, \dots, m-1\}$ we denote by $p_{m|a}$ the partition obtained from p by removing m and inserting the two parts a and $m-a-1$ (if a is 0 or $m-1$ we replace m by $m-1$). If m_l and m_r are two distinct parts of p we denote by $p_{m_l \odot m_r}$ the partition obtained from p

by removing the parts m_l and m_r and inserting m_l+m_r-1. We have $s(p_{m|a}) = s(p)-1$ and $s(p_{m_l\odot m_r}) = s(p)-1$ (notations $p_{m|a}$ and $p_{m_l\odot m_r}$ comes from [Boc80]).

Theorem 2.2. *Let p be an integer partition such that $s(p)+l(p) \equiv 0 \mod 2$. Let m be a part of p, $a \in \{1,\dots,m-2\}$. Set $p' = p\backslash(m)$. Then, we have*

$$\gamma^{std}(m|a,p') = \frac{c(p_{m|a})}{z_{p'}} \qquad \text{and} \qquad \delta^{std}(m|a,p') = \begin{cases} 0 & \text{if } a \equiv 0 \mod 2 \\ \frac{d(p_{m|a})}{z_{p'}} & \text{otherwise} \end{cases}. \tag{2.1}$$

Assume that p has only odd terms. Let m_l and m_r be two distinct parts of p. Set $p' = p\backslash(m_l,m_r)$. Then, we have

$$\gamma^{std}(m_l\odot m_r,p') = \frac{c(p_{m_l\odot m_r})}{z_{p'}} \qquad \text{and} \qquad \delta^{std}(m_l\odot m_r,p') = \frac{d(p_{m_l\odot m_r})}{z_{p'}}. \tag{2.2}$$

The numbers $c(p)$ and $d(p)$ can be interpreted as counting of labeled permutations and $z_{p'}$ as the cardinality of the group which exchanges the labels.

Let $\mathbf{p}$ be a marked profile. We define $\gamma^{irr}(\mathbf{p})$ (resp. $\delta^{irr}(\mathbf{p}) = \gamma^{irr}_1(\mathbf{p}) - \gamma^{irr}_0(\mathbf{p})$) the number of irreducible permutations with given marked profile (resp. the difference between the numbers of irreducible permutations with odd and even spin parity). The below theorem gives recursive formulas for the numbers γ^{irr} and δ^{irr} which involve the numbers γ^{std} and δ^{std}.

Theorem 2.3. *Let $p=(m_1,m_2,\dots)$ be an integer partition such that $s(p)+l(p)\equiv 0 \mod 2$. Let $m\in p$ and $a \in \{0,\dots,m-1\}$ then*

$$\begin{aligned}\gamma^{irr}(m|a,p') =& \gamma^{std}(m+2|m-a,p') \\ & - \sum_{\substack{p'_1\uplus p'_2=p' \\ m_1+m_2=m-1 \\ a_1+a_2=a-1}} \gamma^{irr}(m_1|a_1,p'_1)\ \gamma^{std}(m_2+2|m_2-a_2,p'_2).\end{aligned}$$

Let $(m_l,m_r)\subset p$ then

$$\begin{aligned}\gamma^{irr}(m_l\odot m_r,p') =& \gamma^{std}((m_l+1)\odot(m_r+1),\,p') \\ & - \sum_{p'_1\uplus p'_2=p'} \sum_{\substack{k_1+k_2=m_l-1 \\ 0\le a<k_1}} \gamma^{irr}(k_1|a,\,p'_1)\ \gamma^{std}((k_2+1)\odot(m_r+1),\,p'_2) \\ & - \sum_{p'_1\uplus p'_2=p'} \sum_{\substack{k_1+k_2=m_r-1 \\ 1\le a<k_1-1}} \gamma^{irr}(m_l\odot k_1,\,p'_1)\ \gamma^{std}(k_2+2|a,\,p'_2) \\ & - \sum_{p''\uplus(m)=p'} \sum_{\substack{p_1\uplus p_2=p'' \\ k_1+k_2=m-1}} \gamma^{irr}(m_l\odot k_1,\,p_1)\ \gamma^{std}((k_2+1)\odot(m_r+1),\,p_2)\end{aligned}$$

Moreover, if p has only odd parts we have

$$\begin{aligned}\delta^{irr}(m|a,p') =& (-1)^a\ \delta^{std}(m+2|m-a,p') \\ & - \sum_{\substack{p'_1\uplus p'_2=p' \\ m_1+m_2=m-1 \\ a_1+a_2=a-1}} (-1)^{a_2}\delta^{irr}(m_1|a_1,p'_1)\ \delta^{std}(m_2+2|m_2-a_2,p'_2).\end{aligned}$$

And if $(m_1, m_2) \subset p$ *then*

$$\begin{aligned}\delta^{irr}(m_l \odot m_r, p') =&\delta^{std}(m_l + m_r + 1,\, p') \\ &+ \sum_{p'_1 \uplus p'_2 = p'} \quad \sum_{\substack{k_1+k_2=m_l-1\\ 0\le a<k_1}} \delta^{irr}(k_1|a,\, p'_1)\ \delta^{std}(k_2 + m_r + 1,\, p'_2) \\ &+ \sum_{p'_1 \uplus p'_2 = p'} \quad \sum_{\substack{k_1+k_2=m_r-1\\ 1\le a<k_1-1}} \delta^{irr}(m_l \odot k_1,\, p'_1)\ \delta^{std}(k_2 + 2|a,\, p'_2) \\ &+ \sum_{p'' \uplus (m) = p'} \quad \sum_{\substack{p_1 \uplus p_2 = p''\\ k_1+k_2=m-1}} \delta^{irr}(m_1 \odot k_1,\, p_1)\ \delta^{std}(k_2 + m_r + 1,\, p_2)\end{aligned}$$

where $\delta^{std}(m, p') = \sum_{a=1}^{m-2} \delta^{std}(m|a, p')$.

Theorems 2.3 and 2.2 do not treat the case of Rauzy classes associated to hyperelliptic components $\Omega\mathcal{M}_g^{hyp}(2g-1, 1^k)$ and $\Omega\mathcal{M}_g^{hyp}(g, g, 1^k)$ where (1^k) denotes the partition that contains k times the part 1. The component $\Omega\mathcal{M}_g^{hyp}(2g-1)$ (resp. $\Omega\mathcal{M}_g^{hyp}(g, g)$) corresponds to the extended Rauzy class of the symmetric permutation of degree $2g$ (resp. $2g+1$). We know since [Rau79] that the cardinality of the extended Rauzy class of the symmetric permutation of degree n is $2^{n-1} - 1$. To obtain the cardinality of each hyperelliptic class, we establish a general formula that relates the cardinality of an extended Rauzy class $\mathcal{R}$ associated to a profile p to the one of $\mathcal{R}_0$ obtained from $\mathcal{R}$ by adding k marked points. The extended Rauzy class $\mathcal{R}_0$ has profile $p \uplus (1^k)$.

Theorem 2.4. *Let* k*,* $\mathcal{R}$ *and* $\mathcal{R}_0$ *be as above. Let* r *the number of standard permutations in* $\mathcal{R}$ *then*

$$|\mathcal{R}_0| = \binom{n+k+1}{k}\ |\mathcal{R}| + \binom{n+k}{k-1}\ n\, r.$$

As a particular case of the above theorem, we obtain an explicit formula for the cardinalities of Rauzy classes associated to hyperelliptic components.

Corollary 2.5. *Let* $\mathcal{R} \subset S^o_{2g+k}$ *(resp.* $\mathcal{R} \subset S^o_{2g+k+1}$*) be the extended Rauzy class associated to* $\Omega\mathcal{M}^{hyp}(2g-1, 1^k)$ *(resp.* $\Omega\mathcal{M}^{hyp}(g, g, 1^k)$*) and* $n = 2g$ *(resp.* $n = 2g+1$*), then*

$$|\mathcal{R}| = \binom{n+k+1}{k}\ (2^{n-1} - 1) + \binom{n+k}{k-1}\ n.$$

The paper is organized as follows. In Section 3, we review the definitions of Rauzy classes and extended Rauzy classes. We describe the Rauzy classes of the symmetric permutation $\pi^{sym} \in S^o_n$ defined by $\pi(k) = n - k + 1$ (Section 3.1.3) and the permutation of rotation class $\pi^{rot} \in S^o_n$ defined by $\pi^{rot}(1) = n$, $\pi^{rot}(n) = 1$ and $\pi^{rot}(k) = k$ for $k = 2, \ldots, n-1$ (Section 3.1.3). We recall the classification of Rauzy classes and extended Rauzy classes in terms of connected components of strata of the moduli space of Abelian differential. In particular, we obtain a formula for cardinalities of Rauzy classes in terms of the numbers $\gamma^{irr}(\mathbf{p})$ and $\delta^{irr}(\mathbf{p})$. In section 4, we study standard permutations in order to proove Theorem 2.2. In section 5 we see how standard permutations can be used to describe the set of all permutations and prove Theorems 2.3 and 2.4.

Proofs overview

Now, we explain our strategy to compute cardinalities of Rauzy diagrams.

First we formulate a definition of Rauzy classes in terms of invariants of permutations in Section 3 (see in particular Theorem 3.22). This reformulation follows from the work of [Vee82], [Boi09] and the classification of connected components of strata of Abelian differentials in [KZ03]. Using this geometric definition, we are able to express cardinalities of Rauzy classes in terms of the numbers $\gamma^{irr}(\mathbf{p})$ and $\delta^{irr}(\mathbf{p})$ which counts irreducible permutations with given profile p (see Corollary 3.23).

The computation of the numbers $\gamma^{irr}(\mathbf{p})$ and $\delta^{irr}(\mathbf{p})$ is done in two steps. Both steps use geometrical surgeries used in the classification of connected components of strata [KZ03] and [Lan08]. The first step consists in studying standard permutations. We consider the numbers $c(p)$ and $d(p)$ of labeled permutations and get a recurrence in terms of partitions of $n-1$ for both of them (Theorems 4.12 and 4.18). We then prove that the recurence can be solved into explicit formulas (Theorems 4.13 and 4.19). These explicit formulas corresponds to the ones given in the above introduction. The link between standard permutations and the number of labeled standard permutations as in Theorem 2.2 is proved in Corollary 5.10 and 5.12.

The second step consists in proving Theorem 2.3 which express the numbers $\gamma^{irr}(\mathbf{p})$ (resp. $\delta^{irr}(\mathbf{p})$) in terms of $\gamma^{std}(\mathbf{p})$ (resp. $\delta^{std}(\mathbf{p})$). We use a simple construction: to a standard permutation $\pi \in S_n$ we associate the permutation $\tilde{\pi} \in S_{n-2}$ obtained by "removing its ends". Formally $\tilde{\pi}(k) = \pi(k+1) - 1$ for $k = 1, \ldots, n-2$. The operation $\pi \mapsto \tilde{\pi}$ gives a (trivial) combinatorial bijection between standard permutations in S_n and all permutations in S_{n-2}. As the permutations obtained by this operation are not necessarily irreducible we define Rauzy classes of reducible permutations. To any permutation we can associate a profile and a spin invariant (see Sections 3.3.1 and 3.3.2). As each permutation is a unique concatenation of irreducible permutations, we study how are related the invariants of a permutation to the invariants of its irreducible components (this is done in Lemmas 5.6 and 5.8). In geometric terms, a reducible permutation corresponds to an ordered list of surfaces in which each surface is glued to the preceding and the next one at a singularity. The operation $\pi \to \tilde{\pi}$ can be analyzed as a surgery operation and the invariants of $\tilde{\pi}$ depend only on the ones of π (Proposition 5.9 and 5.11). Theorem 2.3 follows from an inclusion-exclusion counting for irreducible permutations among all permutations.

Contents

3 Permutations, interval exchange transformations and translation surfaces

In this section we define the Rauzy induction of interval exchange transformations on the space of parameters $\mathbb{R}_+^n \times S_n^o$. We study in particular the two combinatorial operations on irreducible permutations $R_t, R_b : S_n^o \to S_n^o$ which define Rauzy classes. Next we recall the relation between translation surfaces and interval exchange transformations. Our aim is to give another definition of Rauzy classes and extended Rauzy classes (Definition 3.4) as well as a classification in terms of invariants of a permutation: the *profile* which is an integer partition, the *hyperellipticity* and the *spin parity* which is an element of $\{0, 1\}$ (Theorem 3.22).

We recall that if p be an integer partition then we denote by $\gamma^{irr}(p)$ (resp. $\gamma_1^{irr}(p)$ and $\gamma_0^{irr}(p)$) the number of irreducible permutations with profile p (resp. profile p and spin parity 1 and 0). We set $\delta^{irr}(p) = \gamma_1^{irr}(p) - \gamma_0^{irr}(p)$. The cardinality of every Rauzy class, but the ones which are associated to components of strata which contain an hyperelliptic component, depend only on the numbers $\gamma^{irr}(p)$ and $\delta^{irr}(p)$ (see Corollary 3.23).

The next two sections of this paper are devoted to the computations of $\gamma^{irr}(p)$ and $\delta^{irr}(p)$. The explicit formulas for the cardinalities of hyperelliptic Rauzy classes are given in Corollary 5.15.

3.1 Rauzy induction and Rauzy classes

3.1.1 Labeled permutations

We introduce a labeled version of permutations which comes from [MMY05] and [Buf06] inspired from [Ker85] (see also [Boi10]). Many constructions are easier to formulate with this definition.

Definition 3.1. A *labeled permutation on a finite set* $\mathcal{A}$ is a couple of bijections $\pi_t, \pi_b : \mathcal{A} \to \{1, \ldots, n\}$ where n is the cardinality of $\mathcal{A}$. The elements of $\mathcal{A}$ are called the *labels* of (π_t, π_b) and $\mathcal{A}$ the *alphabet*.

In order to distinguish labeled permutations from permutations we will sometimes call them *reduced permutations* instead of permutations. The number n is called the *length* of the permutation. To a labeled permutation we associate a reduced one by the map $(\pi_t, \pi_b) \mapsto \pi_b \circ \pi_t^{-1}$. We also consider the natural section given by $\pi \mapsto (id, \pi)$ for which the alphabet of the labeled permutation (id, π) is $\{1, 2, \ldots, n\}$.

A labeled permutation $\pi = (\pi_t, \pi_b)$ is written as a table with two lines

$$\pi = \begin{pmatrix} \pi_t^{-1}(1) & \pi_t^{-1}(2) & \ldots & \pi_t^{-1}(n) \\ \pi_b^{-1}(1) & \pi_b^{-1}(2) & \ldots & \pi_b^{-1}(n) \end{pmatrix}$$

The *top line* (resp. *bottom line*) of π is the ordered list of labels $\pi_t^{-1}(i)$ for $i = 1, \ldots, n$ (resp. $\pi_b^{-1}(i)$ for $i = 1, \ldots, n$). For a reduced permutation π we use the section defined above and write

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi^{-1}(1) & \pi^{-1}(2) & \ldots & \pi^{-1}(n) \end{pmatrix} \quad \text{or simply} \quad \left(\pi^{-1}(1)\ \pi^{-1}(2)\ \ldots\ \pi^{-1}(n)\right).$$

The above notation coincides with the notation of π^{-1} in group theory. With our notation, the label i is at the position $\pi(i)$ on the bottom line. The difference of notation will not cause any problem as we never use the composition of permutations that arises from interval exchange transformations. The only operation considered here is the concatenation (see Section 5.1).

The definitions of standard and irreducible permutations extend to labeled permutations.

Definition 3.2. We say that (π_t, π_b) is *irreducible* (resp. *standard*) if $\pi_b \circ \pi_t^{-1} \in S_n$ is irreducible (resp. standard).

Our aim is to count reduced permutations, however in Section 4 we will mainly deal with labeled ones. In [Boi10], C. Boissy analyze the difference between reduced and labeled permutations.

3.1.2 Rauzy induction and Rauzy classes

Let $T = T_{\lambda,\pi}$ be an interval exchange transformation on $I = [0, a)$ where π is an irreducible labeled permutation on an alphabet $\mathcal{A}$ with n letters and $\lambda \in \mathbb{R}_+^{\mathcal{A}}$ satisfies $\sum_{\alpha \in \mathcal{A}} \lambda_\alpha = a$. For $i = 1, \ldots, n+1$ we set $x_i = \sum_{j=1}^{i-1} \lambda_{\pi_t^{-1}(j)}$ the discontinuities of T and $y_i = \sum_{j=1}^{i-1} \lambda_{\pi_b^{-1}(j)}$ the ones of T^{-1}. We have $x_1 = y_1 = 0$ and $x_{n+1} = y_{n+1} = a$. Let $J = [0, \max(x_n, y_n)) \subset I$. The Rauzy induction of T, denoted by $R(T)$, is the interval exchange $T' = T_{\lambda',\pi'}$ obtained as the first returned map of T on J. The type of T is *top* if $\lambda_{\pi_t^{-1}(n)} > \lambda_{\pi_b^{-1}(n)}$ and *bottom* if $\lambda_{\pi_b^{-1}(n)} > \lambda_{\pi_t^{-1}(n)}$. In the case $\lambda_{\pi_t^{-1}(n)} = \lambda_{\pi_b^{-1}(n)}$ there is no Rauzy induction defined. When T is of type top (resp. bottom) the label $\pi_t^{-1}(n)$ (resp. $\pi_b^{-1}(n)$) is called the *winner* and $\pi_b^{-1}(n)$ (resp. $\pi_t^{-1}(n)$) the *loser*. Let $\alpha \in \mathcal{A}$ (resp. $\beta \in \mathcal{A}$) be the winner (resp. loser) of T. The vector λ' of interval lengths of T' is given by

$$\begin{aligned} \lambda'_\alpha &= \lambda_\alpha - \lambda_\beta, \\ \lambda'_\nu &= \lambda_\nu \quad \text{for all } \nu \in \mathcal{A}\backslash\{\alpha\}. \end{aligned}$$

The permutation $\pi' = R_\varepsilon(\pi)$ is defined as follows, where $\varepsilon \in \{t, b\}$ is the type of T (t for top and b for bottom). Let $\alpha_t = \pi_t^{-1}(n)$ (resp. $\alpha_b = \pi_b^{-1}(n)$) the label on the right of the top line (resp. bottom line). As π is irreducible, the position $m = \pi_b(\alpha_t)$ of α_t on the bottom line (resp. $m = \pi_t(\alpha_b)$ of α_b in the top line) is different from n. We obtain π' from π by moving α_b (resp. α_t) from position n to position $m+1$ in the bottom interval (resp. top interval). The operations R_t and R_b are formally defined by

$R_t(\pi_t, \pi_b) = (\pi_t, \pi'_b)$ where if $m = \pi_b(\alpha_t)$ we have

$$\pi'^{-1}_b(j) = \begin{cases} \pi_b^{-1}(j) & \text{if } j \le m, \\ \pi_b^{-1}(n) & \text{if } j = m+1, \\ \pi_b^{-1}(j-1) & \text{otherwise.} \end{cases}$$

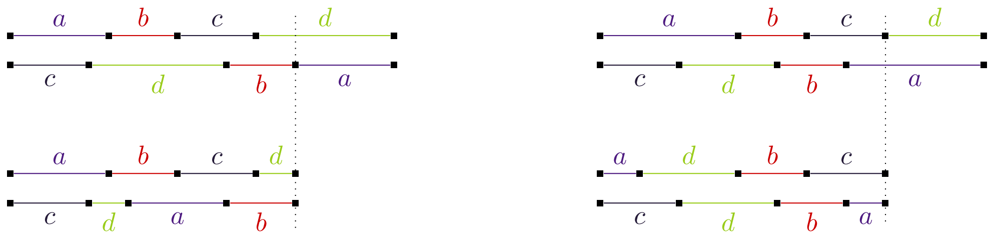
$R_b(\pi_t, \pi_b) = (\pi'_t, \pi_b)$ where if $m = \pi_t(\alpha_b)$ we have

$$\pi'^{-1}_t(j) = \begin{cases} \pi_t^{-1}(j) & \text{if } j \le m, \\ \pi_t^{-1}(n) & \text{if } j = m+1, \\ \pi_t^{-1}(j-1) & \text{otherwise.} \end{cases}$$

The map R_t and R_b are called *Rauzy moves*. An example of a Rauzy induction of an interval exchange transformation is shown in Figure 3.1. The Rauzy moves on reduced permutations are defined using the section $\pi \mapsto (id, \pi)$ and the projection $(\pi_t, \pi_b) \mapsto \pi_b \circ \pi_t^{-1}$ introduced in Section 3.1.1.

$$\text{initial permutation } \pi = \begin{pmatrix} a & b & c & d \\ c & d & b & a \end{pmatrix}$$

$\lambda_a < \lambda_d$: top induction $\qquad$ $\lambda_d < \lambda_a$: bottom induction



$$R_t(\pi) = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} \qquad R_b(\pi) = \begin{pmatrix} a & d & b & c \\ c & d & b & a \end{pmatrix}$$

Figure 3.1: The two alternatives of the Rauzy induction

We consider one more operation called *inversion* and denoted by s which reverses the top and the bottom and the left and the right

$$s \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} = \begin{pmatrix} b_n & \dots & b_2 & b_1 \\ a_n & \dots & a_2 & a_1 \end{pmatrix}.$$

The following is standard.

Lemma 3.3. *The Rauzy moves R_t, R_b and the inversion s preserve irreducible permutations. The Rauzy moves and the symmetry restricted to the set of irreducible permutations are bijections.*

Definition 3.4. Let π be an irreducible permutation. The orbit of π under the action of R_t and R_b (resp. R_t, R_b and s) is called the *Rauzy class* (resp. *extended Rauzy classes*) of π and note it $\mathcal{R}(\pi)$. The *Rauzy diagram* (resp. *extended Rauzy diagram*) of π is the labeled oriented graph with vertices $\mathcal{R}(\pi)$ and edges corresponding to the action of R_t and R_b (resp. R_t, R_b and s).

Let π is a reduced (resp. labeled) permutation than the Rauzy class of π is called a *reduced Rauzy class* (resp. *labeled Rauzy class*).

The standard permutations play a central role in Rauzy classes in particular we have.

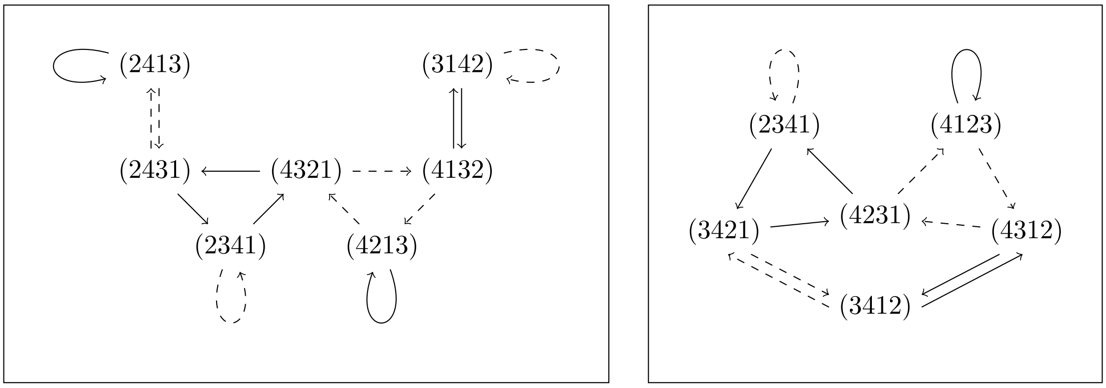
Proposition 3.5 ([Rau79])**.** *Every Rauzy class contains a standard permutation.*

Proof. Let $\mathcal{R}$ be a Rauzy class of permutations on n letters and let $\pi \in \mathcal{R}$. Let $\alpha_t = \pi_t^{-1}(n)$ and $\alpha_b = \pi_b^{-1}(n)$ be the labels of the right extremities. Let $n_b = \pi_b(\alpha_t)$ and $n_t = \pi_t(\alpha_b)$.

If $n_t = \min(n_t, n_b) \neq 1$ then by irreducibility, in the set $\pi_t \circ \pi_b^{-1}(\{n_t + 1, \dots, n\})$ the minimal element is less than n_t. Let n_b' be this minimum and α_b' be the letter for which the minimum is reached. Applying R_t we can move the letter α'_b at the right extremity of the bottom line. After this first step the quantity $n'_b = \min(n_t, n'_b)$ is lesser than $n_t = \min(n_t, n_b)$. For, the case $n_b = \min(n_t, n_b)$, we use R_b to decrease the quantity $\min(n_t, n_b)$.

Iterating succesively R_t or R_b as in the above step, we obtain a permutation such that either $n_t = 1$ or $n_b = 1$. Applying one more time a Rauzy move, we obtain both equal to 1. $\square$

There are only two standard permutations of length 4, $(4\,3\,2\,1)$ and $(4\,2\,3\,1)$, which define two Rauzy classes. Their Rauzy diagrams are presented in Figure 3.2.



(a) The Rauzy diagram of (4321) (b) The Rauzy diagram of (4231)

Figure 3.2: The two Rauzy diagrams of S_4^o.

The labeled rauzy diagrams are coverings of reduced rauzy diagrams (the covering map is the projection $(\pi_t, \pi_b) \mapsto \pi_b$Âă $\circ\, \pi_t^{-1}$). The degree of the covering which gives the multiplicative coefficient between the cardinality of reduced Rauzy classes and labeled Rauzy classes and its computation involves geometric methods which are developed in [Boi10].

3.1.3 Examples of Rauzy diagrams

We denote by π_n^{sym} the *symmetric permutation* on n letters defined by $\pi_n^{sym}(k) = n - k + 1$ for $k = 1, \ldots, n$. In our notation, π_n^{sym} writes

$$\pi_n^{sym} = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}. \tag{3.1}$$

The permutation π_n^{sym} has a Rauzy class which is described in [Rau79] (see also [Yoc05] p. 53).

Proposition 3.6. *The Rauzy class $\mathcal{R}_n^{sym}$ of π_n^{sym} coincide with its extended Rauzy class. It contains $2^{n-1} - 1$ permutations and among them only π_n^{sym} is a standard permutation.*

In that case we remark that the labeled Rauzy class coincide with the reduced one.

We now describe an other class. Let n be a positive integer and let

$$\pi_n^{rot} = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n & 2 & 3 & \ldots & n-1 & 1 \end{pmatrix}. \tag{3.2}$$

The permutation π_n^{rot} is called of *rotation class*. Any interval exchange transformation with permutation π_n^{rot} is a first return map of a rotation. We denote by $\mathcal{R}_n^{rot}$ the Rauzy diagram of π_n^{rot}

We now build a graph $\mathcal{G}_n$. Let $V_n = \{(a,b,c) \in \mathbb{N}^3;\ a,c \geq 1,\ b \geq 0 \text{ and } a+b+c = n\}$. From a triple $(a,b,c) \in V_n$ we define the permutation

$$\pi(a,b,c) = \left(\begin{array}{ccc|ccc|ccc} 1 & \dots & a & a+1 & \dots & a+b & a+b+1 & \dots & a+b+c \\ a+b+1 & \dots & a+b+c & a+1 & \dots & a+b & 1 & \dots & a \end{array}\right). \tag{3.3}$$

Let $\mathcal{G}_n$ be the oriented labeled graph with vertices V_n and edges are of two types

- the *left edges* are $(a,0,c) \to (1,a-1,c)$ and if $b \neq 0$, $(a,b,c) \to (a+1,b-1,c)$,
- the *right edges* are $(a,0,c) \to (a,c-1,1)$ and if $b \neq 0$ $(a,b,c) \to (a,b-1,c+1)$.

From the rules that define the edges, we see that each vertex has exactly one incoming and one outgoing edge of each type. Moreover, in each cycle made by left edges (resp. right edges) there is exactly one element of the form $(a,0,c)$. The number a (resp. c) is the length of the cycle. In Figure 3.3 we draw examples of such graphs.

Proposition 3.7. *The graph $\mathcal{G}_n$ is isomorphic to the Rauzy diagram $\mathcal{R}_n^{rot}$ under the map $(a,b,c) \mapsto \pi(a,b,c)$. The left edges (resp. right edges) in $\mathcal{G}_n^{rot}$ correspond to top (resp. bottom) Rauzy moves in $\mathcal{R}_n^{rot}$.*
Moreover the extended Rauzy diagram of π_n^{rot} has the same set of vertices as $\mathcal{R}_n^{rot}$. The action of s in the extended Rauzy class corresponds to $(a,b,c) \mapsto (c,b,a)$ in $\mathcal{G}_n$.

We remark that for π_n^{rot} the ratio between the cardinalities of labeled and reduced Rauzy classes is $(n-1)!$. This result is a particular instance of a theorem of [Boi10].

Proof. The permutation π_n^{rot} corresponds to the triple $(1,n-2,1) \in V_n$. From the definition of R_t and R_b it can be easily checked that the edges of $\mathcal{G}_n$ corresponds to Rauzy moves on $\pi(a,b,c)$. Hence, the set of permutations associated to V_n is invariant under Rauzy induction. As the graph $\mathcal{G}_n$ is connected, this set is exactly the Rauzy class of π_n^{rot}.

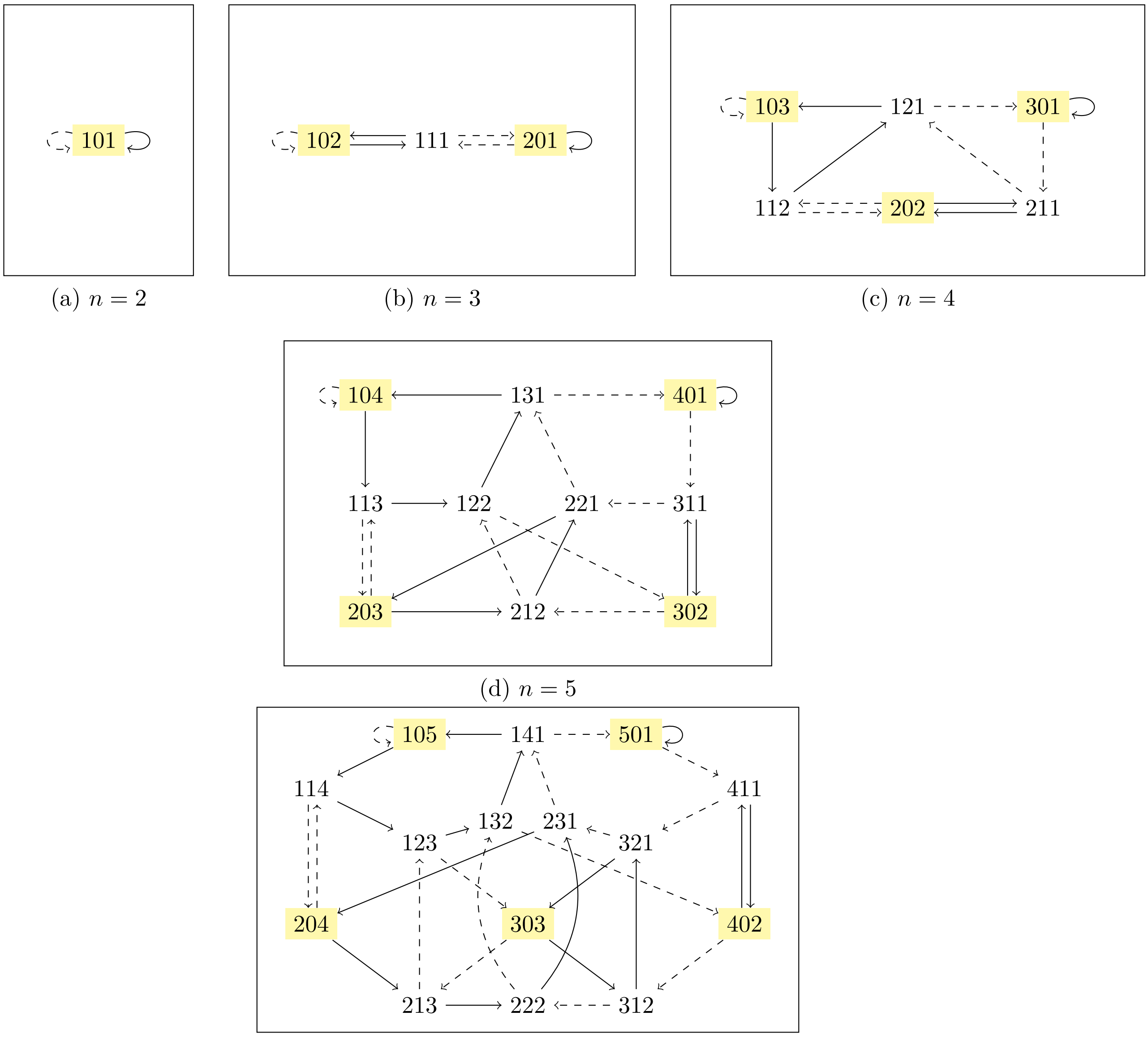
The inversion s exchanges the three parts of the permutation $\pi(a,b,c)$ delimited by the bars in (3.3). The structure of the permutation in three blocks is preserved and we get that $s \cdot \pi(a,b,c) = \pi(c,b,a)$. □

Proposition 3.8. *The Rauzy class $\mathcal{R}_n^{rot}$ of π_n^{rot} coincide with its extended Rauzy class. It contains $\binom{n}{2} = \frac{n(n-1)}{2}$ permutations and among them only π_n^{rot} is a standard permutation.*

3.2 From permutations to translation surfaces

3.2.1 Translation surface

Let S be a compact oriented connected surface. A *translation structure* on S is a flat metric defined on $S - \Sigma$ where $\Sigma \subset S$ is a finite set of points which has trivial holonomy (the parallel transport along a loop is trivial). The latter condition implies that at any point $P \in \Sigma$ the metric has a conical singularity of angle an integer multiple of 2π : the length of a circle centered at a conic point of angle $2\pi m$ with small radius r will not measure $2\pi r$ but $2\pi m r$. More concretely, a translation surface can be built from gluing polygons. Let $P_1, \dots, P_f \subset \mathbb{R}^2$ be a finite collection of polygons and τ a pairing of their sides such that each pair is made of two sides which are parallel, with the same length and opposite normal



(a) $n = 2$ (b) $n = 3$ (c) $n = 4$

(d) $n = 5$

(e) $n = 6$

Figure 3.3: The graphs $\mathcal{G}_n$ for $n = 2, 3, 4, 5, 6$.

vectors. We define the equivalence relation $\sim_\tau$ on the union $P = \cup P_i$: $x_1 \sim_\tau x_2$ if x_1 and x_2 are, respectively, on two sides s_1 and s_2 which are paired by τ and differ by the unique translation that maps s_1 onto s_2. The quotient $S = S(P_i, \tau) := P/\sim_\tau$ is a translation surface for which the metric and the vertical direction are induced from $\mathbb{R}^2$. We call the couple (P_i, τ) a *polygonal representation* of the translation surface S. Reciprocally, any translation surface admits a geodesic triangulation which gives a polygonal representation of the surface.

Let S be a translation surface and $(2\pi n_1, 2\pi n_2, \ldots, 2\pi n_k)$ the list of angles of its conical singularities. The genus g of the surface satisfies

$$2g - 2 = \sum_{i=1}^{k} (n_i - 1). \tag{3.4}$$

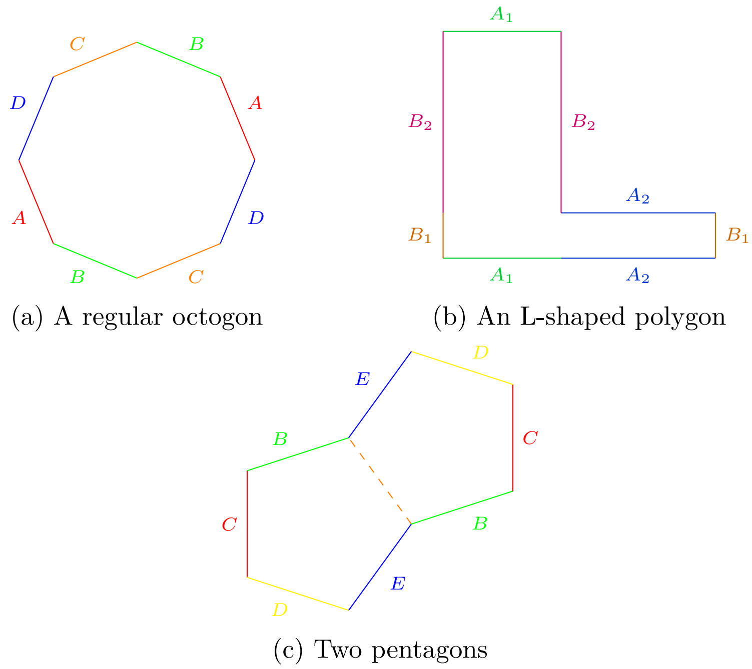


Figure 3.4: Three surfaces built from polygons. The pairings are defined by colors and labels.

The integer partition $p = (n_1, n_2, \ldots, n_k)$ is the *profile* of the translation surface S and Equation (3.4) resumes to $s(p) - l(p) = 2g - 2$ where $s(p) = n_1 + \ldots + n_l$ is the sum of the terms of p and $l(p) = k$ its length. As a consequence the number of even terms in p is even. This is the unique obstruction for a profile of a flat surface: for any integer partition p such that the number of even terms is even there exists a translation surface S with profile p.

The genus, related to the collection of angles in Equation (3.4), can also be deduced from the way the polygons are glued together. Let f (for faces) be the number of polygons. Each pair of sides gives an embedded geodesic segment in the surface, let e (for edges) be the number of those pairs. The vertices of the polygons are identified in a certain number of classes depending on the combinatorics of the pairing τ, let v (for vertices) be the number of classes. Then we have

$$2 - 2g = v - e + f \qquad \text{where } g \text{ is the genus of the surface.} \tag{3.5}$$

Consider the example of Figure 3.4a, the surface obtained from the octogon has four edges and one vertex, thus $2 - 2g = 1 - 4 + 1 = -2$, therefore its genus is 2. On other hand, the angle at the unique conic point of the surface is 6π. The two other examples of Figure 3.4 have the same profile.

If a translation surface has a conical angle of 2π then, from the viewpoint of the metric, the singularity is removable: there exists a unique continuous way to extend the metric at this point. To a surface S with profile $(n_1, n_2, \ldots, n_l, 1, \ldots, 1)$ with k parts equal to 1 we associate a surface $\overline{S}$ with profile $(n_1, n_2, \ldots, n_l)$. We say that surface $\overline{S}$ is obtained from S by marking k points.

3.2.2 Moduli space of translation surfaces

Two translation surfaces S_1 and S_2 are *isomorphic* if there exists an orientation preserving isometry between S_1 and S_2 which maps the vertical direction of S_1 on the vertical direction of S_2. Let $\Omega\mathcal{M}(n_1-1, n_2-1, \ldots, n_k-1)$ be the collection of isomorphism classes of flat surfaces for whose profile is $(n_1, n_2, \ldots, n_k)$. The notation $\Omega\mathcal{M}(\kappa)$ comes from algebraic geometry where $\Omega\mathcal{M}$ is the tangent bundle to the moduli space of complex curves $\mathcal{M}_g$. In this settings, translation surfaces are considered as Riemann surfaces together with an Abelian differential. A conical singularity of angle $2\pi m$ for the flat metric corresponds to a zero of degree $m-1$ of the Abelian differential (see [Zor06] for more details about the relations between flat structure and Abelian differential).

We now define a topology on $\Omega\mathcal{M}$ using the construction with polygons. We first remark that given the combinatorics of polygons (e.g. the cyclic order of the edges in each polygon, and the pairing τ), the set of vectors that are admissible as sides for the polygons forms an open set in $(\mathbb{R}^2)^{e-f+1} = (\mathbb{R}^2)^{2g+s-1}$ where as before v, e and f denote the number of vertices, edges and faces in the polygon. On other hand, two different polygonal representations may give isomorphic translation surfaces. We consider, on polygonal representations, the following operations (see also Figure 3.5)

- The *cut operation* consists in the creation of a new pair of edges between two vertices (if it is possible). This operation creates an edge and the number of faces increases by 1.
- The *paste operation* consists in pasting two polygons along two edges which are paired. This operation delete an edge and the number of faces decreases by 1.

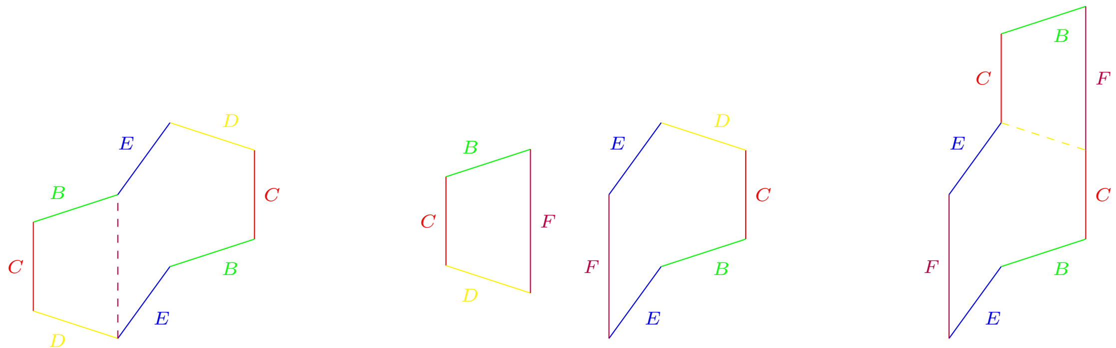


Figure 3.5: An example of one cut followed by one paste.

We have the following.

Proposition 3.9 ([Mas82],[Vee93]). *The isomorphism class of a surface $S(P_i, \tau)$ built from polygons is invariant under cut and paste operations of the polygonal representation (P_i, τ). Moreover, if P and P' are two polygonal representations of the same surface S then there exists a sequence $P_0 = P$, P_1, $\ldots$, $P_n = P'$ of polygonal representations such that P_{i+1} is obtained from P_i either by a cut or a paste operation.*

The above proposition states that the space $\Omega\mathcal{M}$ can be considered as a quotient of a finite union of open sets of $(\mathbb{R}^2)^{2g-2+v}$ by the action of cutting and pasting. The topology of $\Omega\mathcal{M}$ is by definition the quotient topology. As the action of cut and paste operations is discrete, the local system of neighborhood in $\Omega\mathcal{M}$ are open sets in vector spaces. Hence, two translation surfaces are near if they admit decompostions in polygons which have the same combinatorics and roughly the same shape.

We drafted a construction of the moduli space of translation surfaces $\Omega\mathcal{M}$ which is a quotient of the tangent bundle of a Teichmüller space (which corresponds to polygonal representation) by the mapping class group (which corresponds to cut and paste operations). See [Mas82] and the textbooks [Ahl66], [Nag88], [IT92] or [Hub06].

3.2.3 Suspension of a permutation and Rauzy-Veech induction

We recall the method in [Vee82] for building a translation surface from a permutation. The version for labeled permutations comes from [MMY05] and [Buf06]. Let $\pi = (\pi_t, \pi_b)$ be an irreducible labeled permutation, $\mathcal{A}$ its alphabet and $n = |\mathcal{A}|$. A *suspension datum* for π is a collection of vectors $\zeta = (\zeta_\alpha)_{\alpha\in\mathcal{A}} = ((\lambda_\alpha, \tau_\alpha))_{\alpha\in\mathcal{A}} \in (\mathbb{R}_+ \times \mathbb{R})^{\mathcal{A}}$ such that

$$\forall 1 \leq k \leq n-1, \quad \sum_{\pi_t(\alpha)\leq k} \tau_\alpha > 0 \quad \text{and} \quad \sum_{\pi_b(\alpha)\leq k} \tau_\alpha < 0.$$

To each suspension datum ζ we associate a translation surface $S = S(\zeta, \pi)$ in the following way. Consider the broken lines L_t (resp. L_b) in $\mathbb{R}^2$ starting at the origin and obtained by the concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (resp. $\zeta_{\pi_b^{-1}(j)}$) $j = 1, \ldots, n$ (in this order). If the broken line L_t and L_b have no intersection other than the endpoints, we can construct a translation surface S from the polygon bounded by L_t and L_b. The pairing of the sides associate to the side ζ_α of L_t the side ζ_α of L_b (see Figure 3.6). Note that the lines L_t and L_b might have some other intersection points. But in this case, one can still define a translation surface using the *zippered rectangle construction* due to [Vee82]. In the suspension $S = S(\pi, \zeta)$ there is a canonical embedding of the segment $I = [0, |\lambda|)$. The first return map on I of the translation flow of S is the interval exchange map T with permutation π and vector of lengths λ (see Figure 3.6). The Rauzy induction can be

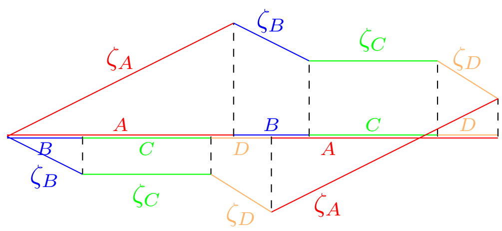


Figure 3.6: A suspension of the permutation $\binom{ABCD}{BCDA}$ and the first return map of the vertical linear flow on its canonical transverse segment.

extended to suspensions and will be still denoted by R. If $\zeta = (\lambda, \tau)$ is a suspension data for π, then $R(\zeta, \pi)$ is the suspension $(\zeta', \pi') = ((\lambda', \tau'), \pi')$ where

– $\pi' = R_\varepsilon(\pi)$ where $\varepsilon \in \{t, b\}$ is the type of T,

– $\zeta'_\alpha = \zeta_\alpha - \zeta_\beta$ where α (resp. β) is the winner (resp. loser) for T.

This extension is known as the *Rauzy-Veech induction*, and is used as a discretization of the Teichmüller flow.

By construction the surfaces $S_{\zeta,\pi}$ and $S_{\zeta',\pi'}$ are isomorphic: the Rauzy-Veech induction corresponds to one cut followed by one paste operations (see Figure 3.7). In particular, by the definition of Rauzy class (Definition 3.4), we have the following proposition which

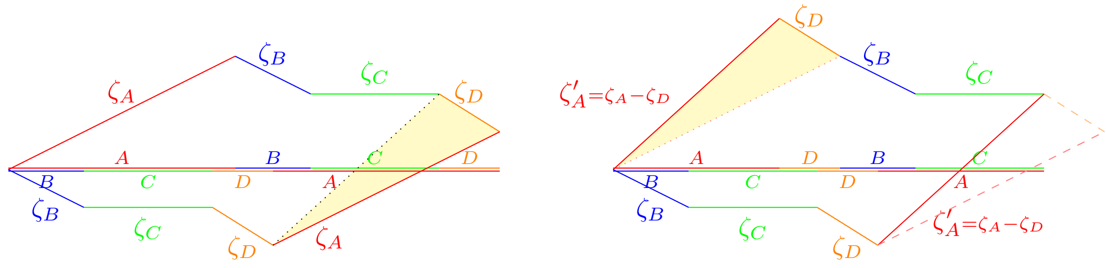


Figure 3.7: The (bottom) Rauzy-Veech induction on a suspension of $\pi = \binom{ABCD}{BCDA}$

is a key ingredient in the correspondance between Rauzy classes and moduli space of translation surfaces.

Proposition 3.10 ([Vee82]). *Let $\mathcal{R}$ be a Rauzy class or an extended Rauzy class. Then, the set of suspensions obtained from permutations in $\mathcal{R}$ is open and connected in $\Omega\mathcal{M}$.*

The case of extended Rauzy class in the above proposition follows from the fact that the involution s on permutations (see Section 3.1.2) can be seen as a central symmetry of the suspension $S(\pi, \zeta)$.

3.3 Permutation invariants of Rauzy classes

We now define the three invariants of permutations that lead to a classification of Rauzy classes.

3.3.1 Interval diagram and profile

Let π be a labeled permutation with alphabet $\mathcal{A}$. We consider a refinement of the permutation σ introduced in [Vee82] which take care of the labels of π. Let $\tilde\sigma$ be the permutation on the set $\overline{\mathcal{A}} \cup \underline{\mathcal{A}} = \{\overline{a};\ a \in \mathcal{A}\} \cup \{\underline{a};\ a \in \mathcal{A}\}$ defined by

$$\tilde\sigma(\overline{a}) = \begin{cases} \overline{\pi_t^{-1}(1)} \quad , & \text{if } \pi_b(a) = 1 \\ \underline{\pi_b^{-1}(\pi_b(a)-1)}, & \text{if } \pi_b(a) \neq 1 \end{cases} \quad \text{and} \quad \tilde\sigma(\underline{a}) = \begin{cases} \overline{\pi_t^{-1}(\pi_t(a)+1)}, & \text{if } \pi_t(a) \neq n \\ \underline{\pi_b^{-1}(n)} \quad , & \text{if } \pi_t(a) = n \end{cases}.$$

Assume that the permutation π is irreducible and consider a suspension S of π. We identify $\overline{a}$ (resp. $\underline{a}$) to the left-half (resp. right-half) of the edge labeled a in S. The permutation $\tilde\sigma$ corresponds to the sequence of half-edges that we cross by turning around vertices of S (see Figure 3.8).

Let π be a labeled permutation on $\mathcal{A}$. We define $\overline{\mathcal{A}_\pi}$ (resp. $\underline{\mathcal{A}_\pi}$) to be the quotient of $\overline{\mathcal{A}}$ (resp. $\underline{\mathcal{A}}$) in which $\overline{\pi_b^{-1}(1)}$ and $\overline{\pi_t^{-1}(1)}$ (resp. $\underline{\pi_t^{-1}(n)}$ and $\underline{\pi_b^{-1}(n)}$) are identified.

Definition 3.11. The *interval diagram* of π is the permutation $\sigma = \sigma_\pi$ on the set $\mathcal{A}_\pi = \overline{\mathcal{A}_\pi} \cup \underline{\mathcal{A}_\pi}$ defined by

$$\sigma_\pi(a) = \begin{cases} \tilde\sigma(\pi_t^{-1}(1)) & \text{if } a = \overline{(\pi_b^{-1}(1), \pi_t^{-1}(1))}, \\ \tilde\sigma(\pi_b^{-1}(n)) & \text{if } a = \underline{(\pi_t^{-1}(n), \pi_b^{-1}(n))}, \\ \tilde\sigma(a) & \text{otherwise.} \end{cases}$$

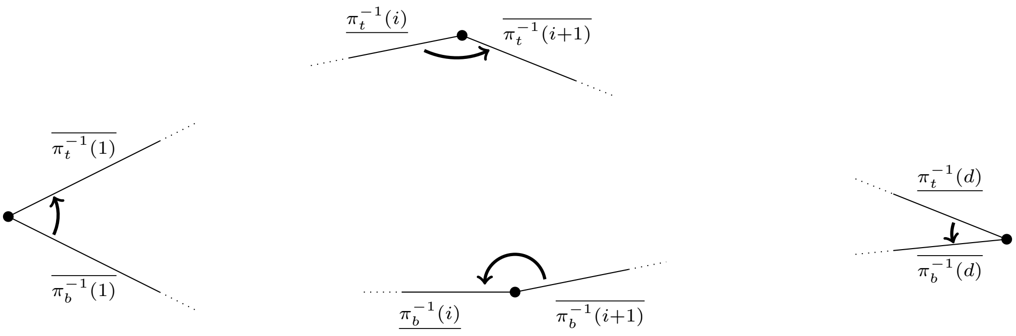


Figure 3.8: The permutation $\tilde{\sigma_\pi}$ and turning around vertices of a suspension of π.

As an example, on the permutation $\left(\begin{smallmatrix} A & B & C & D \\ B & C & D & A \end{smallmatrix}\right)$ in Figures 3.6 and 3.7 the interval diagram is

$$\sigma_\pi = \left(\overline{(B,A)}, \underline{(D,B)}\right) \left(\overline{C}, \underline{B}\right) \left(\overline{D}, \underline{C}\right)$$

The interval diagram σ_π exchanges $\overline{\mathcal{A}_\pi}$ and $\underline{\mathcal{A}_\pi}$. In particular, the permutation σ_π^2 can be written as a product of two permutations $\overline{\sigma_\pi}$ and $\underline{\sigma_\pi}$ on respectively $\overline{\mathcal{A}_\pi}$ and $\underline{\mathcal{A}_\pi}$.

We recall that conjugacy class of permutations of a set with n elements are in bijection with integer partition of n. To a permutation σ we associate the length of the cycles in the disjoint cycle decomposition of σ.

Lemma 3.12. *Let π be an irreducible permutation and S a suspension of π. The profile of S is the integer partition associated to the conjugacy class of the permutation $\overline{\sigma_\pi}$ (or $\underline{\sigma_\pi}$).*

Proof. Following [Vee82] and [Boi10], the permutation $\overline{\sigma_\pi}$ (resp. $\underline{\sigma_\pi}$) can be seen as the crossing of the horizontal direction. In particular each cycle corresponds to a conical singularity of the suspension S and its length k equals the angle divided by 2π. $\square$

3.3.2 The spin parity

Now we define the spin parity of a permutation π whose profile p contains only odd numbers. As the spin parity relies on the classification of quadratic forms over the field with two elements $\mathbb{F}_2$, we first recall this classification in Theorems 3.13 and 3.14. For more details about the spin invariant see [Joh80] and [KZ03].

Let $n \geq 1$ and V a vector space over $\mathbb{F}_2$. A *quadratic form* on V is a map $q : V \to \mathbb{F}_2$ which is an homogeneous polynomial of degree 2 in any coordinate system of V. If q is a quadratic form, then the application B_q defined on $V \times V$ by $B_q(u,v) = q(u+v) - q(u) - q(v)$ is bilinear. The form q is called *nondegenerate* if B_q is nondegenerate. Because the characteristic is two, the form B_q satisfies

$$B_q(u,v) = B_q(v,u) \quad \text{and} \quad B_q(u,u) = 0. \tag{3.6}$$

. If there exists a non degenerate bilinear form B on V which satisifies (3.6) then the dimension of V is even. We consider from now that the dimension $n = 2g$ is even and $V = (\mathbb{F}_2)^{2g}$. On V, there is only one linear equivalence class of nondegenerate bilinear form B that satisfies (3.6). The *standard nondegenerate bilinear* form on $V = \mathbb{F}_2^{2g}$ is the bilinear form B_0 given in coordinates $v = (x_1, y_1, \dots, x_g, y_g) \in V$, $v' = (x'_1, y'_1, \dots, x'_g, y'_g)$ by

$$B_0(v, v') = \sum_{i=1}^{g} (x_i y'_i + x'_i y_i).$$

By the above remark, any non degenerate quadratic form is linearly equivalent to one whose associated bilinear form is B_0. In order to classify quadratic form up to linear equivalence, we assume that q is such that $B_q = B_0$. In other words the quadratic form q writes in terms of the coordinates of v as

$$q(v) = \sum_{i=1}^{n} \left(a_i x_i^2 + b_i y_i^2 + x_i y_i \right), \tag{3.7}$$

where $t = ((a_i, b_i))_{i=1,\dots,n} \in (\mathbb{F}_2)^{2n}$. We denote by q_t the quadratic form (3.7).

Theorem 3.13. *Let $V = (\mathbb{F}_2)^{2n}$ with $n \geq 1$. There are two equivalence classes of non-degenerate quadratic forms over V. They are identified by their Arf invariant $\mathrm{Arf}(q) \in \mathbb{F}_2$ which is defined by*

$$\#\{v \in V; q(v) = 0\} - \#\{v \in V; q(v) = 1\} = (-1)^{\mathrm{Arf}(q)}\ 2^{n-1}.$$

The Arf invariant of the form q_t defined in (3.7) is the number of indices $i \in \{1, \dots, n\}$ such that $(a_i, b_i) = (1, 1)$ modulo 2.

Proof. The proof follows from the cases of $n = 1$ and $n = 2$. For $n = 1$, the form $x^2 + xy + y^2$ is invariant under $Sp(B_0) = GL_2(\mathbb{F}_2)$ whereas the three other forms xy, $x^2 + xy$ and $xy + y^2$ are linearly equivalent. We denote $U_0 = \{(0,0), (0,1), (1,0)\}$ and $U_1 = \{(1,1)\}$ and consider the case of $n = 2$. The case $n = 1$ implies that the forms q_t with $t \in U_0 \times U_0$ are equivalent and using symmetries of coordinates the forms q_t with $t \in U_0 \times U_1 \cup U_1 \times U_0$ are equivalent. There is a linear transformation that maps $q_{(1,1,1,1)}$ to $q_{(0,0,0,0)}$, namely

$$q_{(1,1,1,1)}(x_1 + x_2 + y_2, y_1 + x_1, x_2 + x_1 + y_1, y_2 + x_2) = q_{(0,0,0,0)}(x_1, y_1, x_2, y_2).$$

Hence there are at most two equivalence classes. The fact that we have at least two classes follows from the formula relating the Arf invariant to the number of solutions of $q(v) = 1$. The general case follows by recurrence. $\square$

The formula in Theorem 3.13 states that the Arf invariant of a quadratic form q is the majority value assumed by q on V among 0 and 1. We now states a theorem about the classifcation theorem of all quadratic forms.

Theorem 3.14. *Let $V = (\mathbb{F}_2)^n$ with $n \geq 2$. There are three linear equivalence classes of quadratic forms on V of rank $2g$ with $0 < g < n$:*

- $\{q;\ q|_{\ker(B_q)} \neq 0\}$,
- $\{q;\ q|_{\ker(B_q)} = 0$ *and* $Arf(\overline{q}) = 0\}$ *where* $\overline{q}$ *is* q *on the quotient* $V/\ker(B_q)$,
- $\{q;\ q|_{\ker(B_q)} = 0$ *and* $Arf(\overline{q}) = 1\}$.

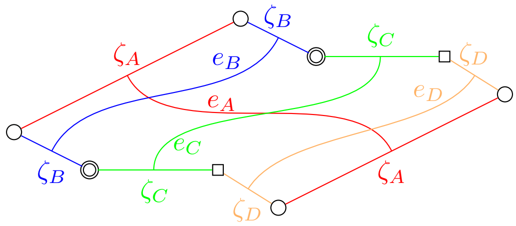


Figure 3.9: Canonic basis of $H_1(S,\Sigma;\mathbb{Z}/2)$ and $H_1(S-\Sigma;\mathbb{Z}/2)$ of a suspension of $\pi = \binom{ABCD}{BCDA}$.

Now, we define the spin parity of a permutation. Let $\pi = (\pi_t,\pi_b)$ be a labeled permutation on the alphabet $\mathcal{A}$ with n elements. Let $V := (\mathbb{F}_2)^{\mathcal{A}}$ and e_α be the elementary vector for which the only non zero coordinate is in position α. The *intersection form* of π is the bilinear form $\Omega = \Omega_\pi$ on V defined by

$$\Omega_{\alpha,\beta} = \Omega(e_\alpha,e_\beta) = \begin{cases} 1 & \text{if } (\pi_t(\alpha)-\pi_t(\beta))(\pi_b(\alpha)-\pi_b(\beta)) < 0, \\ 0 & \text{else.} \end{cases}$$

The matrix Ω corresponds to crossings: the entry (α,β) of the matrix is 1 if and only if the order of $(\pi_t(\alpha),\pi_t(\beta)$ is the opposite of $(\pi_b(\alpha),\pi_b(\beta))$.

Let $S = S(\pi,\zeta)$ be a suspension of π. The sides $(\zeta_\alpha)_{\alpha\in\mathcal{A}}$ of S form a basis of the relative homology $H_1(S,\Sigma;\mathbb{Z}/2)$. The elements (e_α) can be considered as its dual basis in $H_1(S-\Sigma;\mathbb{Z}/2)$ (see Figure 3.9). The intersection form on $H_1(S;\mathbb{Z}/2)$ is well defined on $H_1(S-\Sigma;\mathbb{Z}/2)$ by composition of the natural morphism $H_1(S-\Sigma;\mathbb{Z}/2) \to H_1(S;\mathbb{Z}/2)$ obtained from the inclusion $S-\Sigma \to S$. The matrix Ω corresponds to the the intersection matrix of the vectors $(e_\alpha)_{\alpha\in\mathcal{A}}$ viewed as elements of $H_1(S-\Sigma;\mathbb{Z}/2)$. In particular the rank of Ω_π is $2g$ where g is the genus of the suspension.

We remark that Ω only depends on the topological structure of $S(\pi,\zeta)$ and not on the flat metric. Now, we define a quadratic form q_π. For any closed curve $\gamma : [0,1] \to S$ there is an associated winding number (relative to the flat metric) which is an integer multiple of 2π. We denote by $w(\gamma)$ this integer modulo 2 and extends it by linearity to $H_1(S-\Sigma;\mathbb{Z}/2)$. We may notice that any linear form on $\mathbb{F}_2$ can be canonically transformed into a totally degenerate quadratic form without changing its values as $0^2 = 0$ and $1^2 = 1$. The quadratic form q_π on $H_1(S-\Sigma;\mathbb{Z}/2)$ is

$$q(x) = w(x) + \#(\text{components of x}) + \#(\text{self intersections of x}).$$

Proposition 3.15. *Let π be a permutation. The quadratic form q_π is such that the restriction to $\ker(B_{q_\pi})$ is null if and only if the profile of π as only odd parts.*

Proof. Let q be the quadratic form of π and B its associated bilinear form. The vector space $\ker(B)$ is generated by small loops around the singularities (each loop around a singularity is non trivial in $H_1(S-\Sigma;\mathbb{Z}/2\mathbb{Z})$ and becomes trivial in $H_1(S;\mathbb{Z}/2\mathbb{Z})$). Let γ be a simple curve around a singularity of angle $k2\pi$. The winding number of γ is $w(\gamma) = k$ and hence $q_\pi(\gamma) = k+1$. □

Definition 3.16. Let π be a permutation such that its profile has only odd parts. The *spin parity* of π is the Arf invariant of the quadratic form q_π.

As an example the permutations

$$\pi_0 = \begin{pmatrix} 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ 7\ 4\ 3\ 2\ 1\ 6\ 5\ 0 \end{pmatrix} \quad \text{and} \quad \pi_1 = \begin{pmatrix} 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ 7\ 2\ 1\ 4\ 3\ 6\ 5\ 0 \end{pmatrix}$$

have both profiles (7) but the spin parity are, respectively, 0 and 1. The permutations π_0 and π_1 hence belong to two different Rauzy classes. This fact can be checked by explicit computation of Rauzy classes but is fastidious as the cardinality of Rauzy classes are respectively 5209 and 2327.

3.3.3 Hyperellipticity

A translation surface S is *hyperelliptic* if there exists a morphism of degree two from S to the Riemann sphere $\mathbb{P}^1\mathbb{C}$ such that the flat structure of S comes from a quadratic differential on $\mathbb{P}^1\mathbb{C}$.

Proposition 3.17 ([KZ03])**.** *In the strata $\Omega\mathcal{M}(2g-1)$ (resp. $\Omega\mathcal{M}(g,g)$) there exists a connected component $\Omega\mathcal{M}^{hyp}(2g-1)$ (resp. $\Omega\mathcal{M}^{hyp}(g,g)$) such that each surface in the component is hyperelliptic. These two families are the only connected components of strata without marked point with this property.*

For strata $\Omega\mathcal{M}(2g-1,1^k)$ and $\Omega\mathcal{M}(g,g,1^k)$ which contain marked points, there is also a connected components which comes from the hyperelliptic ones in $\Omega\mathcal{M}(2g-1)$ and $\Omega\mathcal{M}(g,g)$. We will call them *hyperelliptic* as well.

Proposition 3.18 ([KZ03])**.** *Let π be an irreducible permutations with profile $(2g-1)$ or (g,g) and S a suspension of π. Then S is in an hyperelliptic component of $\Omega\mathcal{M}$ defined in Proposition 3.17 if and only if π is in the Rauzy class of a symmetric permutation*

$$\pi_n^{sym} = \begin{pmatrix} 1\ 2\ \dots\ n \\ n\ n-1\ \dots\ 1 \end{pmatrix}.$$

3.4 Definition of Rauzy classes in terms of invariants

As we have seen in Proposition 3.10, we can associate to each Rauzy class and each extended Rauzy class a connected component of a stratum $\Omega\mathcal{M}(p_\pi)$. In this section we recall the results of [Vee82] and [Boi09] which prove how this association can be turned into a one to one correspondance. Next, we explain the classification of connected components of strata of [KZ03] and deduce a classification of Rauzy classes.

3.4.1 Connected components of moduli space and Rauzy classes

In order to get a correspondance between Rauzy classes and connected components of moduli space of translation surfaces, we need to encode a combinatorial data which corresponds to the fact that the Rauzy induction fixes the left endpoint of the interval. Let $\Omega\mathcal{M}(p)$ be a stratum and $m_l \in p$. Let $p' = p\backslash\{m_l\}$. We denote by $\Omega\mathcal{M}(m_l;p')$ the moduli space of translation surfaces $\Omega\mathcal{M}(p)$ with a choosen singularity of degree m_l.

If π is a permutation, we denote by $m_l(\pi)$ the angle of the singularity on the left of π. It corresponds to the length of the cycle of the interval diagram which contains the element $\overline{\pi_b^{-1}(1), \pi_t^{-1}(1)}$ (see Section 3.3.1). To an irreducible permutation we associate a connected component with a choosen singularity of degree m_l.

Theorem 3.19 ([Vee82],[Boi09]). *The association $\pi \mapsto \Omega\mathcal{M}(p_\pi)$ induces a bijection between extended Rauzy classes of irreducible permutations and connected components of strata of moduli spaces $\Omega\mathcal{M}(p)$.*
The association $\pi \mapsto \Omega\mathcal{M}(m_l(\pi); p'_\pi)$ induces a bijection between Rauzy classes and connected components of strata of moduli spaces with a chosen fixed degree.

Corollary 3.20 ([Boi09]). *Let $\mathcal{R}$ be an extended Rauzy class associated to a connected component $\mathcal{C}$ of a stratum $\Omega\mathcal{M}(p)$. Then $\mathcal{R}$ is the union of r Rauzy classes where r is the number of distinct elements of p.*

If $\mathcal{R}$ is an extended Rauzy class, we denote by $\mathcal{R}(m_l)$ the Rauzy class which consist of permutations for which $m_l(\pi) = m_l$. Note that r is not the number of singularities, we have $r = 1$ for any connected component of $\Omega\mathcal{M}(2, 2, 2, 2)$.

There is a map from a component with chosen fixed degree to the one without: $\Omega\mathcal{M}(m_l; p') \to \Omega\mathcal{M}(p)$. At the level of Rauzy classes this corresponds to a disjoint union: the extended Rauzy class corresponding to a permutation π is the union of the Rauzy classes associated to the possible degrees associated to the left endpoint. As an example there is one extended Rauzy $\mathcal{R}$ class with 2638248 elements associated to the connected stratum $\Omega\mathcal{M}(4, 3, 2, 1)$ which is the union of four Rauzy classes $\mathcal{R}(4)$, $\mathcal{R}(3)$, $\mathcal{R}(2)$ and $\mathcal{R}(1)$ with respectively 1060774, 792066, 538494 and 246914 elements.

The labeled Rauzy classes also have a geometric interpretation in terms of moduli space of translation surfaces. If $\pi = (\pi_t, \pi_b)$ is a labeled permutation, then the permutation $\overline{\sigma_\pi}$ deduced from the Rauzy diagram (see Section 3.3.1) is invariant under Rauzy induction which implies a bijection as Theorem 3.19 between labeled Rauzy classes and a moduli space of translation surfaces with combinatorial data. In this case, the combinatorial data consist in a label for each horizontal outgoing separatrices of the surface. The classification of connected component of this moduli space is done in [Boi10]. In particular, he establishes a formula that relates the cardinality of a labeled Rauzy class of a permutation (π_t, π_b) and the cardinality of the reduced Rauzy class of the associated reduced permutation $\pi_b \circ \pi_t^{-1}$. But we emphasize that there is no known relation between labeled extended Rauzy classes and moduli space of translation surfaces.

3.4.2 Kontsevich-Zorich classification of connected components

The strata of moduli spaces of translation surfaces $\Omega\mathcal{M}(p)$ are not connected in general. The three invariants above (profile, spin, and hyperellipticity) as proved in [KZ03] are enough to give a complete classification.

Theorem 3.21 ([KZ03]). *The connected components of a stratum with marked points $\Omega\mathcal{M}(n_1, n_2, \ldots, n_k, 1^l)$ are in bijection with connected components of the stratum $\Omega\mathcal{M}(n_1, n_2, \ldots, n_k)$.*

The classification of connected components of stratum whose profile does not contains any 1 are given by the classification below. For genus $g \geq 4$ we have

– *The strata $\Omega\mathcal{M}(2g - 1)$ and $\Omega\mathcal{M}(g, g)$ with g odd have three components: a hyperelliptic component associated to the symmetric permutations on respectively $2g$ and $2g + 1$ letters. A component with odd spin parity and a component with even spin parity.*
– *The other strata with only odd parts $\mathcal{H}(2m_1 + 1, 2m_2 + 1, \ldots, 2m_n + 1)$ have two connected components which are distinguished by their spin parities.*

– $\Omega\mathcal{M}(g,g)$ *for* g *even has two components: one hyperelliptic and an other one (called the non-hyperelliptic component).*
– *Any other stratum is connected.*

For small genera, the preceding classification holds but there are empty components:
– *genus* 1 *and* 2*: the strata* $\Omega\mathcal{M}(1)$*,* $\Omega\mathcal{M}(3)$ *and* $\Omega\mathcal{M}(2,2)$ *are non empty and connected.*
– *genus* 3*:* $\Omega\mathcal{M}(5)$ *and* $\Omega\mathcal{M}(3,3)$ *have two connected components one hyperelliptic and one odd. The other strata of* $\Omega\mathcal{M}_3$ *are connected.*

By the above theorem, Theorem 3.19 and Theorem 3.19 we obtain the following classification of Rauzy classes.

Theorem 3.22. *Let* $p = (n_1, \ldots, n_k)$ *be an integer partition such that* $s(p) + l(p) \equiv 0 \mod 2$*. Then the set of permutations* π *with profile* p *is the union of* 1*,* 2 *or* 3 *extended Rauzy classes depending on the number of connected components of* $\Omega\mathcal{M}(p)$ *given by Theorem 3.21. Each extended Rauzy class is the union of* r *Rauzy classes where* r *is the number of distinct part in* p*.*

Recall from the introduction that if p be a partition such that $s(p)+l(p) \equiv 0 \mod 2$ we denote by $\gamma^{irr}(p)$ the number of irreducible permutations with profile p. Moreover, if p has only odd terms we denote $\delta^{irr}(p) = \gamma^{irr}_1(p) - \gamma^{irr}_0(p)$ where $\gamma^{irr}_s(p)$ is the number of irreducible permutations with profile p ans spin paruty s.

The below corollary is a direct consequence of Theorem 3.22.

Corollary 3.23. *Let* p *be an integer partition such that* $s(p) + l(p) \equiv 0 \mod 2$ *and* $\Omega\mathcal{M}(p)$ *the stratum of the moduli space of translation surfaces with profile* p*.*

If $\Omega\mathcal{M}(p)$ *is connected then the only Rauzy class* $\mathcal{R}$ *which consists of irreducible permutations with profile* p *satisfies* $|\mathcal{R}| = \gamma^{irr}(p)$*.*

If $\Omega\mathcal{M}(p)$ *is a union of an odd and an even component then there are two Rauzy classes* $\mathcal{R}^{odd}$ *and* $\mathcal{R}^{even}$ *with profile* p *which satisfy* $|\mathcal{R}^{odd}| = \frac{\gamma^{irr}(p)+\delta^{irr}(p)}{2}$ *and* $|\mathcal{R}^{even}| = \frac{\gamma^{irr}(p)-\delta^{irr}(p)}{2}$*.*

If $p = (g,g,1^k)$ *with* g *even, then there are two Rauzy classes* $\mathcal{R}^{hyp}$ *and* $\mathcal{R}^{nonhyp}$ *with profile* p *which satisfy* $|\mathcal{R}^{nonhyp}| = \gamma^{irr}(p) - |\mathcal{R}_{hyp}|$*.*

If $p = (2g-1, 1^k)$ *or* $p = (g,g,1^k)$ *with* g *odd, then there are three Rauzy classes* $\mathcal{R}^{hyp}$*,* $\mathcal{R}^{odd}$ *and* $\mathcal{R}^{even}$ *associated respectively to the hyperelliptic, the odd spin and even spin components of* $\Omega\mathcal{M}(2g-1,1^k)$ *(resp.* $\Omega\mathcal{M}(g,g,1^k)$ *with* g *odd). Then, if* $g \equiv 1,2 \mod 4$ *then*

$$|\mathcal{R}^{odd}| = \frac{\gamma^{irr}(p)+\delta^{irr}(p)}{2} \ - \ |\mathcal{R}_{hyp}| \quad \textit{and} \quad |\mathcal{R}^{even}| = \frac{\gamma^{irr}(p)-\delta^{irr}(p)}{2}.$$

And if $g \equiv 0,3 \mod 4$ *then*

$$|\mathcal{R}^{odd}| = \frac{\gamma^{irr}(p)+\delta^{irr}(p)}{2} \quad \textit{and} \quad |\mathcal{R}^{even}| = \frac{\gamma^{irr}(p)-\delta^{irr}(p)}{2} \ - \ |\mathcal{R}^{hyp}|.$$

As an example, the 461 irreducible permutations on six letters is the union of seven Rauzy classes (respectively five extended Rauzy classes) as below:
– two Rauzy classes (two extended) associated to $\Omega\mathcal{M}_3(5) = \Omega\mathcal{M}_3^{hyp}(5) \cup \Omega\mathcal{M}_3^{odd}(5)$ with respectively 31 and 134 permutations,

- two Rauzy classes (one extended) associated to $\Omega\mathcal{M}_2(3;1,1)$ and $\Omega\mathcal{M}_2(1;3,1)$ with respectively 105 and 66 permutations,
- two Rauzy classes (one extended) associated to $\Omega\mathcal{M}_2(2;2,1)$ and $\Omega\mathcal{M}_2(1;2,2)$ with respectively 90 and 20 permutations,
- one Rauzy class (one extended) associated to $\Omega\mathcal{M}_1(1,1,1,1,1)$ with 15 elements.

Corollary 3.23 can be formulated as well for Rauzy classes introducing natural notations $\gamma^{irr}(m,p')$ and $\delta^{irr}(m,p')$.

4 Enumerating labeled standard permutations

In this section we are interested in the number of standard permutations (Definition 2.1) in any Rauzy class (Definition 3.4) which is the starting point to enumerate the whole class. Recall that the conjugacy classes of S_n are in bijection with integer partition of n. To a permutation σ we associate the integer partition $(n_1, \ldots, n_k)$ whose parts are the lengths of the cycles in the disjoint cycle decomposition. As the bijection is canonic we identify conjugacy classes of S_n and integer partition of n.

Let p be an integer partition and $\sigma \in S_n$ a permutation whose conjugacy class is p. We establish in Proposition 4.1 a bijection between the solutions (τ_t, τ_b) of the equation

$$\sigma = \tau_t\ \tau_b^{-1} \qquad \text{where } \tau_t,\ \tau_b \text{ are } n\text{-cycles of } S_n, \tag{4.1}$$

and the labeled permutations (π_t, π_b) with profile p and fixed labels on outgoing separatrices (see Section 3.3.1). We denote by $c(p)$ the number of solutions of (4.1) as it does not depend on the choice of σ with conjugacy class p. We remark that when p statisfies $l(p) + s(p) \not\equiv 0 \mod 2$ then there is no labeled permutation with profile p (because $s(p) + l(p) = 2g - 2$ where g is the genus of a suspension of π, see (3.4) in Section 3.2). On the other hand, the signature τ of a permutation with conjugacy class p is $\varepsilon(\tau) = (-1)^{s(p)+l(p)}$. Hence, if there is a solution $(\tau_t, \tau_b) \in S_n \times S_n$ of (4.1) the signature of σ is necessarily 1.

If p has only odd parts (in which case the condition $s(p) + l(p) \equiv 0 \mod 2$ is automatic), we denote by $c_1(p)$ (resp. $c_0(p)$) the number of labeled permutations with spin parity 1 (resp. 0) and set $d(p) = c_1(p) - c_0(p)$. Using geometrical analysis, we prove recurrence formulas for c and d (Theorems 4.12 and 4.18) and then provide explicit formulas for both (Theorems 4.13 and 4.19).

4.1 Standard permutations and equations in the symmetric group

The particular form of a standard permutation allows the construction of a surface which is no more built from a polygon but from a cylinder. We explain this construction which can be found in [KZ03], [Zor08] and [Lan08]. Instead of considering a standard permutation π as a double ordering π_t, π_b of the alphabet $\mathcal{A}$, we describe it as a triple of permutations $(\tau_t, \tau_b, \sigma) \in S_{\overline{\mathcal{A}_\pi}} \times S_{\overline{\mathcal{A}_\pi}} \times S_{\overline{\mathcal{A}_\pi}}$ with the following properties

- τ_t and τ_b are $n-1$ cycles,
- $\sigma = \tau_t\ \tau_b^{-1}$ is the permutation $\overline{\sigma_\pi}$,

where the notation $\overline{\mathcal{A}_\pi}$ and $\overline{\sigma_\pi}$ were defined in Section 3.3.1.

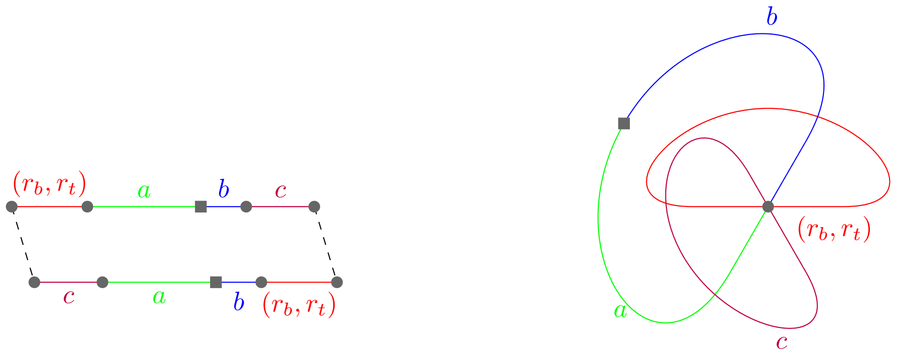
Given $(\tau_t, \tau_b, \sigma) \in S_n \times S_n \times S_n$ with $\tau_t \tau_b^{-1} = \sigma$ we develop the method of [Boc80] which consists in defining another triple $(\tau_t', \tau_b', \sigma') \in S_{n-1} \times S_{n-1} \times S_{n-1}$ in order to relate the solutions of (4.1) in S_n to the ones on S_{n-1}.

4.1.1 Cylindric suspension and equation $\sigma = \tau_t\ \tau_b^{-1}$

Let $\pi = (\pi_t, \pi_b)$ be a labeled standard permutation on the alphabet $\mathcal{A}$ of cardinality $n+1$. Let $r_t = \pi_t^{-1}(1) = \pi_b^{-1}(n)$ and $r_b = \pi_t^{-1}(n) = \pi_b^{-1}(1)$. Let $\zeta \in \mathbb{C}^{\mathcal{A}}$ be such that

- $Im(\zeta_{r_b}) < 0$ and $Re(\zeta_{r_b}) > 0$,
- for all $\alpha \neq r_b$, $Im(\zeta_\alpha) = 0$ and $Re(\zeta_\alpha) > 0$.

Therefore, the vector ζ is not a suspension data as in Section 3.2.3. However, using the same construction with broken lines L_t and L_b, we get a surface which we call a *cylindric suspension* of π (see Figure 4.1). If we glue together the vertical associated to r_b on L_t to the one on L_b we obtain an horizontal cylinder. Its boundary consists of two circles cut in n intervals.



$\tau_t = (r\ a\ b\ c)$
$\tau_b = (r\ c\ a\ b)$

$\overline{\sigma_\pi} = (r\ c\ a)(b)$

Figure 4.1: Cylindric suspension of $\pi = \begin{pmatrix} r_t & a & b & c & r_b \\ r_b & c & a & b & r_t \end{pmatrix}$ and its interval diagram.

There is an arbitrary choice between r_t and r_b as vertical edge. To take care of this flexibility, we label the top and bottom circles with the alphabet

$$\mathcal{A}_\pi = \{(r_b, r_t)\} \cup \{\alpha \in \mathcal{A} : \alpha \neq r_b \text{ and } \alpha \neq r_t\},$$

instead of $\mathcal{A}\backslash\{r_b\}$ (see Figure 4.1). Remark that the labelization of the two circles coincide with the interval diagram defined in Section 3.3.1. Recall that the interval diagram σ_π of π is a permutation defined on the alphabet $\overline{\mathcal{A}_\pi} \cup \underline{\mathcal{A}_\pi}$ which consists in two copies of $\mathcal{A}_\pi$ above. The interval diagram σ_π exchanges $\overline{\mathcal{A}_\pi}$ and $\underline{\mathcal{A}_\pi}$. The square of σ_π decomposes as a product of two permutations $\overline{\sigma_\pi}$ and $\underline{\sigma_\pi}$ on respectively $\overline{\mathcal{A}_\pi}$ and $\underline{\mathcal{A}_\pi}$.

Proposition 4.1. *Let $\mathcal{A}$ be a finite alphabet and r_t, r_b two distinct elements of $\mathcal{A}$. Set $\mathcal{A}' = \{(r_b, r_t)\} \cup \mathcal{A}\backslash\{r_b, r_t\}$. Let $\sigma \in S_{\mathcal{A}'}$, then there is a bijection between the set of labeled standard permutations π on the alphabet $\mathcal{A}$ such that $\overline{\sigma_\pi} = \sigma$ and the set of solutions $(\tau_t, \tau_b) \in S_{\mathcal{A}'} \times S_{\mathcal{A}'}$ such that $\tau_t \tau_b^{-1} = \sigma$.*

Proof. Let n be the cardinality of $\mathcal{A}$. The proof follows directly from the definition of the interval diagram (Definition 3.11). Let π be a standard permutation on $\mathcal{A}$. We associate to π the two n-cycles that consists of the top and bottom lines

$$\tau_t = ((r_b, r_t)\ \pi_t^{-1}(2)\ \pi_t^{-1}(3)\ \ldots\ \pi_t^{-1}(n-1)) \quad \text{and} \quad \tau_b = ((r_b, r_t)\ \pi_b^{-1}(2)\ \pi_b^{-1}(3)\ \ldots\ \pi_b^{-1}(n-1)).$$

The fact that τ_t, τ_b and σ satisfies Equation (4.1) can be resumed in the following picture (Figure 4.2) which represents a vertex of a suspension $S(\pi, \zeta)$ of π together as the action of τ_t, τ_b and σ as permutation. $\square$

An example is shown in Figure 4.1.

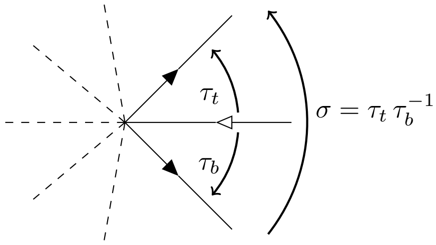


Figure 4.2: The relation $\sigma = \tau_t\, \tau_b^{-1}$ on the level of the interval diagram of π.

Counting labeled stantard permutations is now expressed in a group theoritical way. Let X Y, Z be three conjugacy classes of a finite group G, we want to count the number of solutions of an equation $x\, y\, z = 1$ where $x \in X$, $y \in Y$ and $z \in Z$. This problem is known to be equivalent to a formula involving characters called the Frobenius formula.

Proposition 4.2 (Frobenius formula)**.** *Let G, X, Y and Z as above. Let $N_{X,Y,Z}$ be the number of triples $(x, y, z) \in X \times Y \times Z$ such that $x\ y\ z = 1$. Then*

$$N_{X,Y,Z} = \frac{|X|\ |Y|\ |Z|}{|G|} \sum_{\chi \in \hat{G}} \frac{\chi(X)\ \chi(Y)\chi(Z)}{\chi(1)},$$

where $\hat{G}$ denotes the set of irreducible characters of G.

The proof of Frobenius formula can be found for example in Section 7.2. of [Ser92]. For the numbers $c(p)$ we deduce from Frobenius formula the following expression

$$c(p) = (n-1)! \sum_{\chi \in \hat{S_n}} \frac{\chi(n)^2 \chi(p)}{\chi(1)}. \tag{4.2}$$

It is a hard to pass from expression (4.2) which involves characters to a formula which involves numbers. The recursive construction we adopt does not use Frobenius formula. However there are some works, for example [GS98] (see Theorem 4.16), that from Frobenius formula obtain formulas for the value of $c(p)$. The conjugacy class of σ encodes the stratum associated to the suspension of π. For the numbers $d(p) = c_1(p) - c_0(p)$, there is still an approach using Group Theory. The spin parity can be viewed as a refinement of the signature of a permutation in the Sergeev group [EOP08].

4.1.2 Recursive construction

In order to obtain formulas for the numbers $c(p)$ and $d(p)$ we follow an approach of [Boc80]. Let $\mathcal{A}$ be an alphabet of size n and $\sigma \in S_{\mathcal{A}}$ a permutation. Let (τ_t, τ_b, σ) be a solution of Equation (4.1) and $x \in \mathcal{A}$. Starting from a triple $(\tau_t, \tau_b, \sigma) \in S_{\mathcal{A}}$ of equation (4.1), we choose a letter $x \in \mathcal{A}$, then we remove x in both cycles τ_t and τ_b and get two $(n-1)$-cycles τ'_t and τ'_b on $\mathcal{A}' = \mathcal{A}\backslash\{x\}$. Set $\sigma' = \tau'_t\ \tau'^{-1}_b$, we want to know the relation between σ and σ'.

The $(n-1)$-cycles τ'_t and τ'_b obtained are formally given by

$$\tau'(y) = \begin{cases} \tau(x) & \text{if } y = \tau^{-1}(x), \\ \tau(y) & \text{otherwise.} \end{cases} \qquad \text{where } \tau \text{ equals } \tau_t \text{ or } \tau_b.$$

The operation $\tau \mapsto \tau'$ can be obtained as a multiplication by a transposition, where we consider τ' as a permutation on $\mathcal{A}$ which fixes x. More precisely

$$\tau_t' = (x\ \tau_t(x))\ \tau_t \quad \text{and} \quad \tau_b'^{-1} = \tau_b^{-1}\ (x\ \tau_b(x)). \tag{4.3}$$

To τ_t' and τ_b' which are $(n-1)$-cycles on $\mathcal{A}'$ we associate the permutation σ' by the formula $\sigma' = \tau_t'\ \tau_b'^{-1}$. Using formulas (4.3) we write σ' as a product involving (τ_t, τ_b, σ) and the letter x

$$\sigma' = \tau_t' \tau_b'^{-1} = (x\ \tau_t(x))\, \sigma\, (x\ \tau_b(x)). \tag{4.4}$$

The conjugacy class of σ' depends only of the positions of x and $\tau_t(x)$ in the cycle decomposition of σ. If $p = (n_1, \dots, n_k)$ and $p' = (n_1', \dots, n_{k'}')$ are integer partitions we denote $p \uplus p' = (n_1, \dots, n_k, n_1', \dots n_{k'}')$ their disjoint union. If m is an integer we write $m \in p$ if m is a part of p and if q is an integer partition we write $q \subset p$ if there exists p' such that $p = q \uplus p'$. In which case p' is denoted $p \backslash q$.

Definition 4.3 ([Boc80])**.** Let p be an integer partition of n. Let $m \in p$ and $a \in \{1, 2, \dots, m-2\}$, we call the *splitting* of m in p by a the integer partition

$$p_{m|a} = (a,\, m-a-1)\ \uplus\ p \backslash \{m\}.$$

Let $(m_l, m_r) \subset p$ we call the *collapsing* of m_l and m_r in p the integer partition

$$p_{m_l \odot m_r} = (m_l + m_r - 1)\ \uplus\ p \backslash (m_l, m_r).$$

Remark that if p is a partition of n then both $p_{m|a}$ and $p_{m_l \odot m_r}$ are partitions of $n-1$.

Proposition 4.4 ([Boc80])**.** *Let (τ_t, τ_b, σ), p the conjugacy class of σ, $x \in \mathcal{A}$ and $(\tau_t', \tau_b', \sigma')$ be as above. If x and $\tau_t(x)$ are in the same cycle of σ with length m, then the conjugacy class of σ' is $p_{m|a}$ where a is the smallest number such that $\sigma^a(\tau_t(x)) = x$. If x and $\tau_t(x)$ are in different cycles of σ of length, respectively, m_l and m_r, then the profile of σ' is $p_{m_l \odot m_r}$.*

We remark that x, $\tau_t(x)$ and $\tau_b(x)$ belong to the same cycle c of σ. More precisely, $\tau_b(x)$ and $\tau_t(x)$ are successive letters in c, as by definition $\sigma(\tau_b(x)) = \tau_t(x)$.

Proof of 4.4. By (4.3) and (4.4), the differences between σ and σ' occur for $\sigma^{-1}(x)$ and $\tau_b(x)$ for which we have

$$\sigma'(\tau_b(x)) = \sigma(x) \qquad \text{and} \qquad \sigma'(\sigma^{-1}(x)) = \tau_t(x). \tag{4.5}$$

We first prove the first part of the proposition. We assume that x and $\tau_t(x)$ belong to different cycles c_l and c_r of σ whose lengths are, respectively, m_l and m_r. We write $c_l = (x\ A\ \sigma^{-1}(x))$ and $c_r = (\tau_t(x)\ B\ \tau_b(x))$ where A and B are two blocks of labels which may be empty. The cycles c_l and c_r collapse in σ' in a unique cycle c $=(\tau_t(x)\ B\ \tau_b(x)\ A\ \sigma^{-1}(x))$. Because x is removed the length of c is $m_l + m_r - 1$.

Now, consider the second part of the proposition. We assume that x and $\tau_t(x)$ are in the same cycle c of σ of length m. Because $\sigma(\tau_b(x)) = \tau_t(x)$, the cycle of σ containing x writes $c = (\tau_t(x)\ A_t\ \sigma^{-1}(x)\ x\ \sigma(x)\ A_b\ \tau_b(x))$, with $\sigma(x) \neq \tau_b(x)$ and $\sigma^{-1}(x) \neq \tau_t(x)$. As before, A_t and A_b are two blocks which may be empty. Now σ' has the same cycle decomposition as σ but the cycle containing x splits into two cycles $c_t = (\tau_t(x)\ A_t\ \sigma^{-1}(x))$ and $c_b = (\sigma(x)\ A_b\ \tau_b(x))$. The lengths a_t and a_b of the cycles c_t and c_b can be defined symmetrically by $\sigma^{a_t}(\tau_t(x)) = x$ and $\sigma^{-a_b}(\tau_b(x)) = x$. Therefore, as the label x is removed, those lengths satisfy the expression $a_t + a_b = m - 1$. $\square$

Proposition 4.4 is the heart of the recurrence formula for the numbers $c(p)$ (Theorem 4.12).

4.2 Spin parity

Let $(\tau_t,\, \tau_b\, \sigma) \in S_{\mathcal{A}} \times S_{\mathcal{A}} \times S_{\mathcal{A}}$ be a solution of (4.1) and $x \in \mathcal{A}$. The suppression of the label x in the cycle decomposition of τ_t and τ_b studied in the preceding section leads to a solution $(\tau'_t,\, \tau'_b,\, \sigma')$ on $\mathcal{A}\backslash\{x\}$. Let S be a cylindric suspension of (τ_t, τ_b, σ). The geometric operation associated to the suppression of x corresponds to remove a cylinder associated to the edge ζ_x in S (see Figure 4.3). The operation leads to a cylindric suspension S' of $(\tau'_t, \tau'_b, \sigma')$. Proposition 4.4 can be interpreted as an answer to the stratum behavior of the operation $(S, \zeta_x) \mapsto S'$ (see Proposition 4.6). In this section, we analyze the geometric operation and get a relation between the spin parities of S and S'.

4.2.1 Removing a cylinder in a translation surface

In a cylindric suspension S of a triple $(\tau_t, \tau_b, \sigma) \in S_{\mathcal{A}} \times S_{\mathcal{A}} \times S_{\mathcal{A}}$, a label $x \in \mathcal{A}$ corresponds to a horizontal geodesic ζ_x in S which join two singularities (possibly the same). More generally, let S be a translation surface and ζ a geodesic segment between two singularities of S. We assume that ζ contains no singularity in its interior. Such a segment is called a *saddle connection*.

Definition 4.5. Let S be a translation surface and ζ a saddle connection in S. A geodesic cylinder which contains ζ in its interior and each of its boundary circle contains an endpoint of ζ and no other singularity is called a *cylinder associated to* ζ.

In the case of cylindric suspension each edge ζ_x is a saddle connection and there are several cylinders that are associated to ζ_x but we emphasize that in general given a saddle connection in a translation surface there is no associated cylinder. Let S be a cylindric suspension whose permutations are defined on the alphabet $\mathcal{A}$. The cylinders associated to an edge ζ_x which are of interest for our purpose are cylinders for which the boundary circles are obtained by a straight line in the polygonal representation joining the endpoints of ζ_x in the bottom circle to the endpoints of ζ_x in the bottom circle as in the left part of Figure 4.3.

Let S be a translation surface, ζ a saddle connection in S, C a cylinder associated to ζ and c_1, c_2 its boundary circles. Denote by S' the surface which is obtained from S by removing the interior of C and identifying c_1 and c_2 under the unique isometry $f : c_1 \to c_2$ that maps the endpoint of ζ in c_1 to the endpoint of ζ in c_2. In the surface S' there is a saddle connection c' which corresponds to the identified boundary circles c_1 and c_2 in S. The operation $(S, C) \to S'$ is invertible as soon as we know the saddle connection c' in S' and the parameters of the cylinder C which is removed in S, namely its height $h \in (0, \infty)$ and a twist parameter $\theta \in S^1$. The converse operation $(S', c', h, \theta) \to S$ is called *bubbling a handle* in [KZ03] and *a figure eight operation* in [EMZ03].

Consider a triple $(\tau_t,\, \tau_b,\, \sigma) \in S_{\mathcal{A}} \times S_{\mathcal{A}} \times S_{\mathcal{A}}$ satisfying (4.1) and an associated cylindric suspension S. Let ζ_x be the edge in S associated to a label x in $\mathcal{A}$ and C an associated cylinder whose boundary circles are straight line in the polygonal representation as in Figure 4.3. The surface S' obtained by removing the cylinder C is still a cylindric suspension but of the triple $(\tau'_t, \tau'_b, \sigma')$ obtained by removing x in the cycle decomposition of τ'_t and

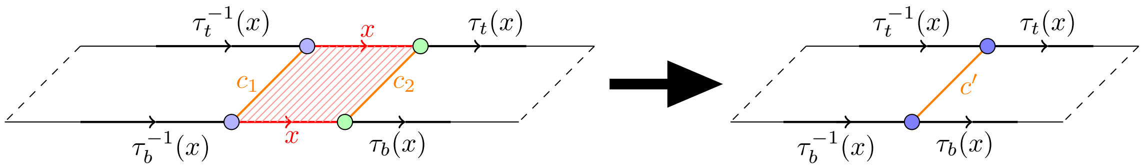


Figure 4.3: Removing the cylinder associated to ζ_x in a cylindric suspension.

τ_b' as defined Section 4.1.2. While the choice of a cylinder associated to x is not unique, the surface S' is. With our convention, the set of outgoing edges of each singularity P of S is invariant under the permutation σ. The cycle c of σ containg x corresponds to the startpoint of ζ_x while the endpoint of ζ_x corresponds to the cycle of σ containing $\tau_t(x)$. Proposition 4.4 can then be rephrased in terms of translation surfaces, cylinders and strata.

Proposition 4.6. *Let $S \in \Omega\mathcal{M}(n_1, \dots, n_k)$ be a translation surface, ζ a saddle connection in S and C a cylinder associated to ζ. Let S' be the surface obtained by removing the cylinder C in S. If the endpoints of ζ corresponds to the same singularity of degree κ_1 in S and the start and end of ζ are separated by an angle $(2a+1)\pi$ then the stratum of S' is $\Omega\mathcal{M}(a, n_1 - a - 1, n_2, \dots, n_k)$. If the endpoints of ζ corresponds to two different singularities of S of degrees respectively κ_1 and κ_2 the the stratum of S' is $\Omega\mathcal{M}(n_1 + n_2 - 1, n_3, \dots, n_k)$.*

Let S be a translation surface, $\Sigma \subset S$ its singularities, ζ a saddle connection in S and C a cylinder of S associated to ζ. Let S' be the translation surface obtained by removing C from S, $\Sigma' \subset S'$ its singularities, and c' the saddle connection in S' which corresponds to the identified boundary circles c_1 and c_2 of C. We define a map $\Psi : H_1(S' \backslash \Sigma'; \mathbb{Z}/2) \to H_1(S \backslash \Sigma; \mathbb{Z}/2)$ which will be used to compare the spin parities of S and S'.

Recall that the surgery operation $S \mapsto S'$, does not affect $S \backslash C$. Hence, if $\xi \subset S$ is a closed curve disjoint from the cylinder C, it defines a curve $\xi' \subset S'$. Let $\xi' \subset S'$ is a closed curve which intersects c'. We assume that the intersection is transverse. Let $\xi \subset S$ be the curve which coïncides with ξ' outside of C and, for each intersection P' of ξ' and c', we replace P' by the unique geodesic segment in C which joins the preimages $P_1 \in c_1$ and $P_2 \in c_2$ of P' and do not intersect ζ.

Lemma 4.7. *Let S , Σ, S' and Σ' as above. Then the map $\xi' \mapsto \xi$ defines a map $\Psi : H_1(S' \backslash \Sigma'; \mathbb{Z}/2) \to H_1(S \backslash \Sigma; \mathbb{Z}/2)$. Moreover Ψ is injective, preserves the intersection forms and the winding numbers.*

Proof. The map $\xi' \mapsto \xi$ is well defined on homology because it preserves boundaries. Let $\xi' \subset S' \backslash \Sigma'$ be a simple closed curve such that $[\xi] = 0 \in H_1(S \backslash \Sigma; \mathbb{Z}/2)$. Then there is a disc $D \subset S \backslash \Sigma$ such that $\xi = \partial D$. The disc D goes down to a disc in S' and shows that $[\xi'] = 0$.

If ξ' is disjoint from c', it is clear that the intersection with ξ' is preserved and $w'(\xi') = w(\xi)$. Now if ξ' is transverse to c' then the pieces added to build ξ are all parallel and in particular do not intersect and has no winding. As the preceding case, the intersection with ξ' is preserved and $w'(\xi') = w(\xi)$. $\square$

4.2.2 Spin parity in the collapsing case

Let S be a translation surface with spin parity. Depending on the alternative of Proposition 4.6, the behavior of the spin structure is different. Let ζ be a saddle connection in S whose endpoints are two different singularities of S, C a cylinder associated to ζ and S' the surface obtained by removing the cylinder C in S. The genus of S is the same as the one of S' and we have the following result.

Lemma 4.8. *Let S, ζ, C and S' as above. Then S' has a spin parity and is the same as the one of S.*

Proof. From Proposition 4.6, we know that if S has a spin structure (meaning that all its singularities have degrees even multiples of 2π) then S' has also one. Recall that the spin structure of, respectively, S and S' are given by Arf invariants of quadratic forms q_S and $q_{S'}$ on $H_1(S;\mathbb{Z}/2)$ and $H_1(S';\mathbb{Z}/2)$ (see Section 3.3.2).

Let $\Psi : H_1(S'\backslash\Sigma';\mathbb{Z}/2) \to H_1(S\backslash\Sigma;\mathbb{Z}/2)$ be the map of Lemma 4.7. As all singularities of S and S' are conical angles of odd multiple of 2π the winding numbers $w : H_1(S\backslash\Sigma;\mathbb{Z}/2) \to \mathbb{Z}/2$ and $w' : H_1(S'\backslash\Sigma';\mathbb{Z}/2) \to \mathbb{Z}/2$ are well defined on $H_1(S;\mathbb{Z}/2)$ and $H_1(S';\mathbb{Z}/2)$. In the collapsing case, the genus of S equals the genus of S' and hence the vector spaces $H_1(S;\mathbb{Z}/2)$ and $H_1(S';\mathbb{Z}/2)$ have the same dimension.

As Ψ is injective, it is an isomorphism. $\overline{\Psi}$ preserves the intersection form and the winding number, thus $q_{S'} = q_S \circ \overline{\Phi}$ and the Arf invariant of $q_{S'}$ and q_S are equal. This proves that S and S' have the same spin parity. □

4.2.3 Spin parity in the splitting case

We now consider the case of a translation surface S with a saddle connection ζ which has the same singularity $P \in S$ as endpoints. Let C be a cylinder associated to ζ. By Proposition 4.6, removing C in S gives a surface S' whose genus is the one of S' minus 1. The start and the end of the geodesic ζ form an angle at the point P which is an odd multiple of π that we denote $(2a+1)\pi$ (see Proposition 4.4 and Proposition 4.6). In order to get the recurrence for the numbers $d(p)$, we have two cases to treat:

– S and S' have a spin parity, which corresponds to a odd (Lemma 4.9),

– S has a spin parity but S' has not, which corresponds to a even (Lemma 4.10).

Similarly to Lemma 4.8, we have.

Lemma 4.9. *Let S and C as above. We assume that S has a spin parity and that a is odd. Then S' obtained by removing C in S has a spin parity and is the same as the one of S.*

Proof. We consider the maps Ψ and $\overline{\Psi}$ of Lemma 4.7. The map $\overline{\Psi}$ identifies a subspace of codimension 2 of $H_1(S;\mathbb{Z}/2)$ with $H_1(S';\mathbb{Z}/2)$. Let c be a circumference of the cylinder C. Then, the symplectic complement of $H_1(S';\mathbb{Z}/2)$ in $H_1(S;\mathbb{Z}/2)$ is the subspace $M = \mathbb{Z}/2\ [c] \oplus \mathbb{Z}/2\ [\zeta]$. Hence $q_S \simeq q'_S \oplus q_S|_M$ and, as the Arf invariant is additive, to compare the Arf invariant of q_S and $q_{S'}$ we compute the Arf invariant of $q_S|_M$.

As ζ is geodesic and its start and end are separated by an angle $(2a+1)\pi$ we have $w(\zeta) = a \mod 2$ and hence $q_S([\tilde{\zeta}]) = a+1 = 0 \mod 2$. On other hand $q_S([c]) = 1$, and from Theorem 3.13 we get that $Arf(q_S|_M) = 0$. Thus q_S and $q_{S'}$ have the same Arf invariant which proves that S and S' have the same spin parity. □

Now, we treat the case of a even. The surface S' obtained after removing the cylinder has no spin but the surface S can have one. In the following lemma the surface S' is fixed and we count how many surfaces of each spin parity we get by the procedure of adding a cylinder. Let $(\tau'_t, \tau'_b, \sigma')$ the combinatorial datum associated to a cylindric suspension S'. We assume that the profile p of S' contains only odd numbers excepted two, m_t and m_b and we write $p = (m_t, m_b) \uplus q$. Let P_t and P_b be the two singularities of S' of conical angles respectively m_t and m_b. We fix a vertex v_t corresponding to P_t in the top circle of S'. Consider all saddle connections that joins v_t to a vertex associated to P_b in the bottom line of the circle of S' (see Figure 4.4). The following is similar to Lemma 14.4 of [EMZ03].

Lemma 4.10. *Let S', $P_t, P_b \in S'$, m_t, m_b and v_t as above. Then there are m_b vertices in the bottom circle of S' associated to P_b. Amongst the m_b cylindric suspension obtained by adding a cylinder to $(S', [vt, v_b])$ where v_b is a vertex associated to P_b in the bottom line, half of them have an odd spin parity and half of them have an even spin parity.*

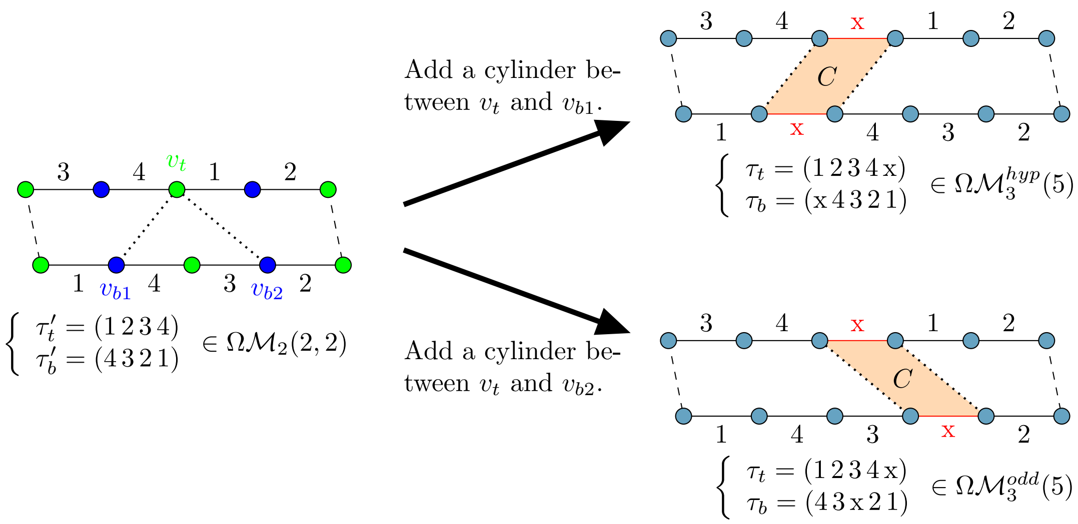


Figure 4.4: The two ways of adding a cylinder to a cylindric suspension in $\Omega\mathcal{M}(2,2)$.

Proof. There are exactly m_b vertices associated to P_b in the bottom circle as the conical angle at P_b is $2\pi m_b$. We fix v_b associated to P_b in the bottom cylinder. We use the same strategy as in Lemma 4.9, we use a map $H_1(S';\mathbb{Z}/2) \to H_1(S;\mathbb{Z}/2)$ and then look at the symplectic complement of its range.

Consider a small neighborhoods V_b of P_b in S' and c the saddle connection that joins v_t to v_b. Any other saddle connection between v_t and a representative of P_b in the bottom circle can be obtained by adding to c an arc of circle contained in V_b. Hence each curves that joins v_t to a representative of P_b in the bottom line can be numeroted with respect to the angle from c. We denote them by $c_0 = c, c_1, \ldots, c_{m_b-1}$. Let S_j, $j = 0, \ldots, m_b-1$, be the surface obtained by adding a cylinder corresponding to c_j and q_j its associated quadratic form. The contribution of the module $M_j = \mathbb{Z}/2[c_j] \oplus \mathbb{Z}/2[x_j] \subset H_1(S;\mathbb{Z}/2)$ to the spin structure is $q_j(c_j) = q(c) + j \mod 2$ and $q_j(x_j) = 1$. In particular $Arf(q_j) = Arf(q) + j \mod 2$ which proves the lemma. $\square$

4.3 Formulas for $c(p)$ and $d(p)$

In this section we prove formulas for the numbers $c(p)$ and $d(p)$. We will use two notations for partitions of an integer n. Either $p = (n_1, n_2, \ldots, n_k)$ where $n_1, \ldots, n_k$ are positive integers whose sum are n. Or $p = (1^{e_1}, 2^{e_2}, \ldots, n^{e_n})$ where e_i denotes the number of times i occurs in p. The numbers e_i satisifies $\sum e_i\ i = n$.

4.3.1 Marked points

We first consider the presence of 1 in the integer partition p. They correspond to marked point in the associated cylindric suspension S. See for example Figure 4.1 where the vertex represented by a square with outgoing edge b is a marked point.

Proposition 4.11. *We have $c((1^n)) = d((1^n)) = (n-1)!$ and, more generally, if p is a partition of the integer n and k is a non negative integer then*

$$c(p \uplus (1^k)) = \frac{(n+k-1)!}{(n-1)!}\ c(p).$$

If moreover p has only odd parts, then

$$d(p \uplus (1^k)) = \frac{(n+k-1)!}{(n-1)!}\ d(p).$$

Proof. The identity permutation $1 \in S_n$ is the only element with profile (1^n). On the other hand, the solutions of the form $(\tau_t, \tau_b, 1)$ of Equation (4.1) are given by $(c, c, 1)$ where c is any n-cycles. Thus $c((1^n)) = (n-1)!$ is the number of n-cycles in S_n. As the partition (1^n) corresponds to a torus (a surfaces with genus 1), it is well known that the spin is odd. Hence $d((1^n)) = c((1^n))$. More generally, adding marked points in a surface do not modify the spin parity.

We denote by C_n the set of n-cycles in S_n. Let p be a partition of n and $\sigma' \in S_n$ whose conjugacy class is p. Let

$$E' = \{(\tau'_t, \tau'_b) \in C_n \times C_n |\ \tau'_t\ {\tau'_b}^{-1} = \sigma'\}.$$

Let $\sigma \in S_{n+1}$ be the permutation which equals σ' on $\{1, \ldots, n\}$ and such that $\sigma(n+1) = (n+1)$ and

$$E = \{(\tau_t, \tau_b) \in C_{n+1} \times C_{n+1} |\ \tau_t\ \tau_b^{-1} = \sigma\}.$$

We claim that there is a canonic bijection $E \to E' \times \{1, \ldots, n\}$. The conclusion of the lemma follows from the claim which we prove now.

The map $E \to E'$ on the first factor correspond to remove $(n+1)$ in the cycles τ_t and τ_b as in Section 4.1.2. The map $E \to \{1, \ldots, n\}$ on the second factor is $(\tau_t, \tau_b) \mapsto {\tau_t}^{-1}(n+1)$. As $\sigma(n+1) = n+1$, we have ${\tau_t}^{-1}(n+1) = {\tau_b}^{-1}(n+1)$. The preimage (τ_t, τ_b) of the element $(\tau'_t, \tau'_b, x) \in E' \times \{1, \ldots, n\}$ is given by

$$\tau(i) = \begin{cases} n+1 & \text{if } i = x, \\ \tau(x) & \text{if } i = n+1, \\ \tau(i) & \text{otherwise,} \end{cases} \qquad \text{for } \tau = \tau_t \text{ or } \tau = \tau_b.$$

$\square$

4.3.2 Two formulas for $c(p)$

We first give a recurrence formula for the number $c(p)$ of labeled standard permutations in the stratum associated to p. The initialization $c((1)) = 1$ of the recurrence can be considered as a particular case of Proposition 4.11.

Theorem 4.12 ([Boc80] prop. 4.2.)**.** *Let $p = (n_1, \ldots, n_k)$ be a partition of an integer $n \geq 2$, then*

$$c(p) = \sum_{i=2}^{k} n_i \; c(p_{n_1 \odot n_i}) + \sum_{a=1}^{n_1-2} c(p_{n_1|a}).$$

Proof. Let $\sigma \in S_n$ whose conjugacy class is p such that the length of the cycle containing n is n_1. As in the proof of Proposition 4.11 we set

$$E(\sigma) = \{(\tau_t, \tau_b) \in C_n \times C_n; \tau_t \; \tau_b^{-1} = \sigma\}.$$

To an element $(\tau_t, \tau_b) \in E$ we associate $(\tau_t', \tau_b', \tau_t(n)) \in C_{n-1} \times C_{n-1} \times \{1, \ldots, n-1\}$ where (τ_t', τ_b') is obtained from (τ_t, τ_b) by removing n in their cycle decomposition (see Section 4.1.2). The map $E \to C_{n-1} \times C_{n-1} \times \{1, \ldots, n-1\}$ is injective. As proved in Proposition 4.4, the conjugacy class of $\sigma' = \tau_t' \; \tau_b'^{-1}$ depends on the nature of the cycle of σ that contains $\tau_t(n)$. The formula of the theorem follows by summing over all possibilities for $\tau_t(n)$. The first sum corresponds to the cases where $\tau_t(n)$ is in a different cycle from the one of n. The second sum corresponds to the cases where n and $\tau_t(n)$ are in the same cycle. □

Boccara in [Boc80] find an explicit formula from the recurrence of Theorem 4.12 using an identity involving a polynom and integration.

Theorem 4.13 ([Boc80])**.** *Let $p = (n_1, n_2, \ldots, n_k)$ be a partition of the integer n. Then, we have*

$$c(p) = \frac{2(n-1)!}{n+1} \left(\sum_{q \subset (n_2, n_3, \ldots, n_k)} (-1)^{s(q)-l(q)} \binom{n}{s(q)}^{-1} \right).$$

From the theorem, we deduce several explicit values

Corollary 4.14. *Let $n = 2k+1$ then*

$$c((n)) = \frac{2 \; (n-1)!}{n+1}.$$

Let $n = n_1 + n_2 \equiv 0 \mod 2$, then

$$c((n_1, n_2)) = \frac{2 \; (n-1)!}{n+1} \; \frac{\binom{n}{n_1, n_2} + (-1)^{n_1+1}}{\binom{n}{n_1, n_2}}.$$

We also have

Proposition 4.15. *Let k be a positive even integer then*

$$c((2^k)) = \frac{(2k-1)!}{2(k-1)(k+1)}.$$

Using the representation theory of the symmetric group A. Goupil and G. Schaeffer [GS98] gave an explicit formula for more general numbers than $c(p)$. Their formula has the advantage of containing only positive numbers. In our particular case we get

Theorem 4.16 ([GS98])**.** *Let p be a partition of the integer n with length k. We set $g = (n-k)/2$. Then we have*

$$c(p) = \frac{z_p}{2^{2g}} \sum_{g_1+g_2=g} \frac{(2g_1)!}{2g_1+1} \binom{n-1}{2g_1} S_{k,g_2}(p),$$

where $S_{k,g} \in \mathbb{Q}[x_1, x_2, \ldots, x_k]$ is the symmetric polynomial

$$S_{k,g}(x_1, x_2, \ldots, x_k) = (k+2g-1)! \sum_{(p_1,p_2,\ldots,p_k) \models g} \prod_{i=1}^{k} \frac{1}{2p_i+1} \binom{x_i - 1}{2p_i},$$

where $(p_1, \ldots, p_k) \models g$ design the set of k-tuples $(p_1, \ldots, p_k)$ of non-negative integers whose sum is g. And z_p is the cardinality of the centralizer of any permutation in the conjugacy class associated to p. Writing $p = (1^{e_1}, 2^{e_2}, \ldots, n^{e_n})$ in exponential notation we have

$$z_p = \prod_{i=1}^{n} e_i!\ i^{e_i}.$$

In [Wal79], D. Walkup made a conjecture about the asymptotic behavior of the numbers c which was proved few years later by R. Stanley in [Sta81].

Theorem 4.17 ([Wal79],[Sta81])**.** *Let $(p_i)_{i\geq 0}$ be a sequence of partition of integers $(n_i)_{i\geq 0}$ such that n_i tends to infinity and the number of 1 in p_i is $O(\log(n_i))$ then*

$$c(p_i) \sim 2(n_i-2)!(1+o(1)).$$

The asymptotic behavior of the above theorem proves that in Boccara's formula (Theorem 4.13) the only contribution comes from the factor $\frac{2(n-1)!}{n+1}$ and the sum in parentheses is asymptotically $(1+o(1))$. For the particular cases in Corollary 4.14 this fact is clear.

4.3.3 A formula for $d(p)$

For an integer partition p whose parts are odd numbers, recall that $c_1(p)$ and $c_0(p)$ denote the number of standard permutations with fixed labels and respectively odd and even spin parity. We have $c(p) = c_1(p) + c_0(p)$ and $d(p) = c_1(p) - c_0(p)$. As for c, we first prove a recurrence formula and then solve it explicitely.

The recurrence formula is similar to Theorem 4.12.

Theorem 4.18. *Let $p = (n_1, \ldots, n_k)$ be an integer partitions with odd parts then*

$$d(p) = \sum_{i=2}^{k} n_i\ d(p_{n_1 \odot n_i}) + \sum_{\substack{a=1 \\ a \equiv 1 \mod 2}}^{n_1-2} d(p_{n_1|a}).$$

Proof. The proof is identic to the one of Theorem 4.12. We fix a permutation p and an element $\sigma \in S_n$ such that the conjugacy class of σ is p. We assume that the cycle containing n has length n_1.

Let E_s be the set of standard permutations (τ_t, τ_b) with labels σ and spin parity $s \in \{0, 1\}$. According to the position of $\tau_t^{-1}(n)$ we separate E_s in different subsets.

If $\tau_t^{-1}(n)$ and n are in different cycles, then we apply the Lemma 4.8 and we get that their number is

$$\sum_{j \neq i}^{k} n_j c_s(p_{n_i \odot n_j}).$$

If n and $\tau_t^{-1}(n)$ are in the same cycle, then we differentiate the case a odd and a even (see Section 4.2.1). For a even, Lemma 4.9 gives that the total number of such standard permutation is

$$\sum_{a \equiv 1\,[2]} c_s(p_{n_1|a}).$$

For a odd, Lemma 4.10 implies that their number is

$$\frac{1}{2} \sum_{a \equiv 0\,[2]} c(p_{n_1|a}).$$

As this last term does not depend on the spin parity s, it cancels in the difference $c_1(p) - c_0(p)$. $\square$

The formula for the numbers $d(p)$ is given by the following.

Theorem 4.19. *Let p be an integer partition with only odd parts, then the number $d(p)$ depends only on the sum n and the length k of p. Set $g = (n-k)/2$ then*

$$d(p) = \frac{(n-1)!}{2^g}.$$

Proof. Set $\tilde{d}(n, k) := (n-1)!/2^{(n-k)/2}$. Those numbers satisfy the recurrences

$$\tilde{d}(n+1, k+1) = n\, \tilde{d}(n, k) \quad \text{and} \quad \tilde{d}(n+1, k-1) = \frac{n}{2}\, \tilde{d}(n, k).$$

On the other hand if $i, j \in \{1, \ldots, k\}$ and $a \in \{1, \ldots, n_i - 2\}$, we have for the sum $s(p_{m_i \odot m_j}) = s(p_{m_i|a}) = s(p) - 1$ and for the length $l(p_{n_i \odot n_j}) = l(p) - 1$ and $l(p_{n_i|a}) = l(p) + 1$. It is then straightforward to check that $\tilde{d}$ satisfies the same recurrence as the formula given in Theorem 4.18. The initial value needed to start the recurrence is the one for the only partition of 1 which is $p = (1)$. But $\tilde{d}(1, 1) = 1 = d((1))$. Hence $d(p) = \tilde{d}(s(p), l(p))$ for all partitions with odd parts. $\square$

5 From standard permutations to cardinality of Rauzy classes

We now prove a recurrence formula for the numbers $\gamma^{irr}(p)$ (resp. $\delta^{irr}(p) = \gamma_1^{irr}(p) - \gamma_0^{irr}(p)$) in terms of the number of standard permutations $\gamma^{std}(p)$ (resp. $\delta^{std}(p) = \gamma_1^{std}(p) - \gamma_0^{std}(p)$). We relate the latter ones to the numbers $c(p)$ and $d(p)$ computed in the preceding section. The recurrence formula is based on the construction of suspensions for any permutation (non necessarily irreducible) and a geometrical analysis of the concatenation of permutations.

5.1 Irreducibility, concatenation and non connected surfaces

5.1.1 Concatenation and irreducible permutations

Let π_1 (resp. π_2) be a labeled permutation on the alphabet $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). The *concatenation* $\pi_1 \cdot \pi_2$ is the labeled permutation on the disjoint union $\mathcal{A}_1 \sqcup \mathcal{A}_2$ defined by

$$\begin{pmatrix} a_1 & \dots & a_{n_1} \\ b_1 & \dots & b_{n_1} \end{pmatrix} \cdot \begin{pmatrix} a'_1 & \dots & a'_{n_2} \\ b'_1 & \dots & b'_{n_2} \end{pmatrix} = \begin{pmatrix} a_1 & \dots & a_{n_1} & a'_1 & \dots & a'_{n_2} \\ b_1 & \dots & b_{n_1} & b'_1 & \dots & b'_{n_2} \end{pmatrix}.$$

The concatenation of two reduced permutations can be defined from the section $\pi \mapsto (id, \pi)$ and projection $(\pi_t, \pi_b) \mapsto \pi_b \circ \pi_t^{-1}$ (see Section 3.1.1). More precisely, let π_1 and π_2 be two reduced permutations of lengths n_1 and n_2. The concatenation $\pi = \pi_1 \cdot \pi_2$ is the permutation of length $n_1 + n_2$ defined by

$$\pi(i) = \begin{cases} \pi_1(i) & \text{if } 1 \leq i \leq n_1, \\ \pi_2(i - n_1) + n_1 & \text{if } n_1 + 1 \leq i \leq n_1 + n_2. \end{cases}$$

One has the following elementary.

Proposition 5.1. *A permutation $\pi \in S_n$ is irreducible if and only if it can not be written as a non trivial concatenation.*

Each (reduced or labeled) permutation has a unique decomposition in irreducible permutations.

As an example, we write in the table below the decomposition of the reducible permutations of length 4. We call *class* of a permutation π the ordered list of the lengths of the irreducible components of π (which is a *composition* of 4, e.g. an ordered list of positive integers whose sum is sum 4).

permutation	decomposition	class
(1234)	$(1)\cdot(1)\cdot(1)\cdot(1)$	$[1,1,1,1]$
(1243)	$(1)\cdot(1)\cdot(21)$	$[1,1,2]$
(1324)	$(1)\cdot(21)\cdot(1)$	$[1,2,1]$
(2134)	$(21)\cdot(1)\cdot(1)$	$[2,1,1]$
(2143)	$(21)\cdot(21)$	$[2,2]$
(1342)	$(1)\cdot(231)$	$[1,3]$
(1423)	$(1)\cdot(312)$	$[1,3]$
(1432)	$(1)\cdot(321)$	$[1,3]$
(2314)	$(231)\cdot(1)$	$[3,1]$
(3124)	$(312)\cdot(1)$	$[3,1]$
(3214)	$(321)\cdot(1)$	$[3,1]$

As a corollary, we get a formula relating factorial numbers $n! = |S_n|$ to $p(n) = |S_n^o|$.

Corollary 5.2. *Let $f(n)$ be the number of irreducible permutations in S_n. Then*

$$n! = \sum_{k=1}^{n} \sum_{c_1+\ldots+c_k=n} f(c_1)\, f(c_2)\, \ldots\, f(c_k), \tag{5.1}$$

5.1.2 Suspensions of reducible permutations

Let π_1 and π_2 be two labeled permutations on the alphabet $\mathcal{A}$ of lengths respectively n_1 and n_2 and $\pi = \pi_1 \cdot \pi_2 = (\pi_t, \pi_b)$ their concatenation of length $n = n_1 + n_2$. If $\zeta \in \mathbb{C}^{\mathcal{A}}$ then

$$\sum_{1 \leq j \leq n_1} \zeta_{\pi_t^{-1} j} = \sum_{1 \leq j \leq n_1} \zeta_{\pi_b^{-1} j}. \tag{5.2}$$

Thus there is no suspension data for π (see Section 3.2.3). But if π_1 and π_2 are irreducible, we can assume that n_1 is the only index such that (5.2) holds.

Definition 5.3. Let π be a labeled permutation on the alphabet $\mathcal{A}$ and $\pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_k$ its decomposition in irreducible permutations. Let $\mathcal{A}_j$ be the alphabet of π_j. A *suspension data* for π is a vector in $\zeta \in \mathbb{C}^{\mathcal{A}}$ such that each $(\zeta_\alpha)_{\alpha \in \mathcal{A}_j}$ is a suspension data for the irreducible permutation π_j.

In the case of the irreducible permutation on 1 letter $\pi = \binom{A}{A}$, the suspension datum is an element $\zeta_A \in \mathbb{R}_+ \times i\mathbb{R} \subset \mathbb{C}$.

Let π and ζ as in the above definition. Then, as for suspension of irreducible permutations in Section 3.2.3, we build two broken lines L_t and L_b made, respectively, of the concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ and $\zeta_{\pi_b^{-1}(j)}$. The surface obtained by identifying the side ζ_α on L_t with the side ζ_α on L_b is a sequence S_1, S_2, …, S_k of translation surfaces such that S_j and S_{j+1} are connected at a singularity. In the case of the degenerate permutation $\binom{A}{A}$ the surface associated to $\zeta_A \in \mathbb{R}_+ \times \mathbb{R}$ corresponds to a (degenerate) sphere with two conical singularities of angle 0. We take as convention that the stratum of $\binom{A}{A}$ is $\Omega\mathcal{M}(0,0)$ (see Figure 5.1).

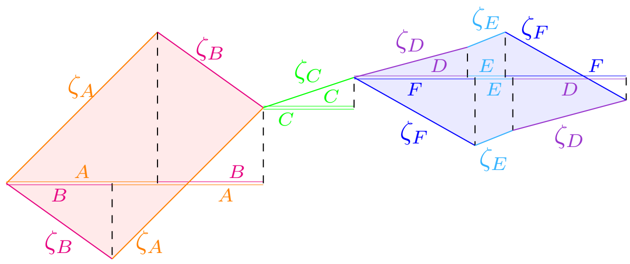


Figure 5.1: A suspension of the reducible permutation $\binom{ABCDEF}{BACFED} = \binom{AB}{BA} \cdot \binom{C}{C} \cdot \binom{DEF}{FED}$.

5.1.3 Marking of a permutation

Let π_1 and π_2 be two permutations. We want to deduce the profile of the permutation $\pi = \pi_1 \cdot \pi_2$ as defined in Section 3.3.1 from the profiles of π_1 and π_2. We first look at an example with the following permutations

$$\pi_1 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,5\,4\,2\,1 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,5\,4\,1\,3 \end{pmatrix}. \tag{5.3}$$

Both permutations have have profile $(3,1)$ but the products $\pi_1 \cdot \pi_1$, $\pi_1 \cdot \pi_2$, $\pi_2 \cdot \pi_1$ and $\pi_2 \cdot \pi_2$ have respectively profiles $(5,3,1)$, $(7,1,1)$, $(3,3,3)$ and $(5,3,1)$. In a product $\pi_1 \cdot \pi_2$ the permutations are glued at the right of π_1 and the left of π_2. To keep track of left and right, we consider profile of permutation with an additional data which encodes the configuration of the two singularities at both extremities of the permutation. In the introduction, we defined markings in term of suspension. We give here a more combinatorial version based on the interval diagram of a permutation defined in Section 3.3.1.

Definition 5.4. Let π be a permutation, Γ its interval diagram and c_l (resp. c_r) be the cycle in Γ that corresponds to the left (resp. right) endpoint of π.

If $c_l = c_r$, let m be the length of c_l and $2a$ be the number of edges in Γ between the outgoing edge on the left of π and the incoming edge on the right of π. The *marking* of π is the couple (m, a) which we call a marking of the *first type* and denote by $m|a$.

If $c_l \neq c_r$, let m_l and m_r be respectively the lengths of c_l and c_r. The *marking* of π is the couple (m_l, m_r) which we call a marking of the *second type* and denote by $m_l \odot m_r$.

The notation similar to the one in Definition 4.3 is explained by the Corollaries 5.10 and 5.12 below.

For the permutations π_1 and π_2 defined in (5.3) the interval diagrams are respectively

$$\Gamma(\pi_1) = (\overline{(3,1)}\ \underline{2})\ (\underline{3}\ \overline{4}\ \underline{(5,1)}\ \overline{2}\ \underline{4}\ \overline{5})) \quad \text{and} \quad \Gamma(\pi_2) = (\overline{3}\ \underline{1}\ \overline{(2,1)}\ \underline{4}\ \overline{5}\ \underline{2})\ (\overline{4}\ \underline{(5,3)}).$$

Hence the markings are respectively $1 \odot 3$ and $3 \odot 1$. Examples of a marking of the first type with profile $(3,1)$ are given by the permutations

$$\pi_3 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,4\,5\,1\,3 \end{pmatrix} \quad \pi_4 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 4\,5\,3\,2\,1 \end{pmatrix} \quad \pi_5 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,5\,3\,4\,1 \end{pmatrix}$$

The interval diagrams and markings of the above permutations are respectively

$$\begin{aligned}
\Gamma(\pi_3) &= (\overline{(2,1)}\ \underline{(5,3)}\ \overline{4}\ \underline{2})\ \overline{3}\ \underline{1})(\overline{5}\ \underline{4}) && \text{with marking } 3|0,\\
\Gamma(\pi_4) &= (\overline{(4,1)}\ \underline{2}\ \overline{3}\ \underline{(5,1)}\ \overline{2}\ \underline{3}\ )\ (\overline{5}\ \underline{4}) && \text{with marking } 3|1,\\
\Gamma(\pi_5) &= (\overline{(2,1)}\ \underline{4}\ \overline{5}\ \underline{2}\ \overline{3}\ \underline{(5,1)})\ (\underline{3}\ \overline{4}) && \text{with marking } 3|2.
\end{aligned}$$

Let p be an integer partition. The markings that occur in a permutation π with profile p are

- the markings $m|a$ where $m \in p$ and $a \in \{0,\dots,m-1\}$,
- the markings $m_1 \odot m_2$ where $m_1, m_2 \in p$ and $m_1 \neq m_2$,
- the markings $m \odot m$ for m which appears at least twice in p.

We remark that for a permutation π with marking of the first type $m|a$ the number a belongs to $\{0,\dots,m-1\}$ whereas for a standard permutation a belongs to $\{1,\dots,m-2\}$.

Definition 5.5. Let π be a permutation with profile p and marking $m|a$ (resp. $m_l \odot m_r$). The *marked profile* of p is the couple $(m|a, p')$ (resp. $(m_l \odot m_r, p')$) where p' is the integer partition $p \backslash (m)$ (resp. $p \backslash (m_l, m_r)$).

We naturally extend the definition of γ, γ^{irr}, γ^{std}, δ, δ^{irr} and δ^{std} to marked profiles.

5.1.4 Profile and spin parity of a concatenation $\pi_1 \cdot \pi_2$

We now answer to the question asked previously about the profile of a concatenation. The lemma below expresses the marked profile of a concatenation in terms of the marked profiles of its irreducible components.

Lemma 5.6. *Let π_1 and π_2 be two permutations and let $\pi = \pi_1 \cdot \pi_2$ be their concatenation. The following array shows how deduce the marked profile of π from the marked profiles of π_1 and π_2.*

marked profile for π_1	*marked profile for π_2*	*marked profile for π*
$(m\|a,\ p')$	$(n\|b,\ q')$	$(m+n+1\|a+b,\ p' \uplus q')$
$(m\|a,\ p')$	$(n_l \odot n_r,\ q')$	$(m+n_l+1 \odot n_r,\ p' \uplus q')$
$(m_l \odot m_r,\ p')$	$(n\|b,\ q')$	$(m_l \odot m_r+n+1,\ p' \uplus q')$
$(m_l \odot m_r,\ p')$	$(n_l \odot n_r,\ q')$	$(m_l \odot n_r,\ p' \uplus q' \uplus (m_r+n_l+1)$

.

In particular, a concatenation $\pi_1 \cdot \pi_2$ has a marking of the first type if and only if both of π_1 and π_2 have a marking of the first type.

Proof. Let Γ (resp. Γ_1 and Γ_2) be the interval diagram of π (resp. π_1 and π_2). Let c_1 (resp. c_2) be the cycle associated to the right of π_1 (resp. the left of π_2). The diagram Γ is built from the disjoint union of Γ_1 and Γ_2 by gluing the cycles c_1 and c_2. More precisely, let

$$\pi_1 = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} x'_1 & \dots & x'_{n'} \\ y'_1 & \dots & y'_{n'} \end{pmatrix}.$$

Then $c_1 = \left(\underline{(x_n, y_n)}\ A\right)$ and $c_2 = \left(\overline{(y'_1, x'_1)}\ A'\right)$ where A and A' design blocks of letters. In the concatenation $\pi = \pi_1 \cdot \pi_2$, the cycles c_1 and c_2 are glued into $c = \left(\overline{x'_1}\ A'\ \overline{y'_1}\ \underline{y_n}\ A\ \underline{x_n}\right)$.

Hence, the length of $|c|$ is $|c_1|+|c_2|+1$. In particular the profile of p of π can be computed from the profiles p_1 and p_2 of respectively π_1 and π_2 as $p = (p_1\backslash(|c_1|)) \uplus (p_2\backslash(|c_2|)) \uplus (|c_1|+|c_2|+1)$. We have proved how the profile of a concatenation $\pi = \pi_1 \cdot \pi_2$ can be deduced from the profiles and markings of its components π_1 and π_2. We now consider the marking of the permutation π.

We treat only the case of two markings of type one, the other being similar. We keep the notation Γ_1, Γ_2, c_1 and c_2 as above. The cycle c_1 (resp. c_2) of Γ_1 (resp. Γ_2) which corresponds to the right of π_1 (resp. the left of π_2) can be written as $c_1 = \left(\overline{(y_1, x_1)}\ A\ \underline{(x_n, y_n)}\ B\right)$ (resp. $c_2 = \left(\overline{(y'_1, x'_1)}\ A'\ \underline{(x'_{n'}, y'_{n'})}\ B'\right)$) where A, B, A' and B' are blocks of letters. The angles in the marking are $a = |A|$ and A'. Those two cycles become one in π which is

$$\left(\overline{(y_1, x_1)}\ A\ \underline{x_n}\ \overline{x'_1}\ A'\ \underline{(x'_{n'}, y'_{n'})}\ B'\ \overline{y'_1}\ \underline{y_n}\ B\right).$$

The angle a (resp. b) in the marking of π_1 (resp. π_2) is the length of A (resp. A') divided by 2. The structure of the cycle c shows that the angle in the marking of π is the length of $A\ x_n\ x'_1\ A'$ divided by 2 which equals $a + b + 1$. □

Now, we consider the spin parity of a permutation whose profile contains only odd parts. We would like to have a lemma similar to Lemma 5.6 which relates profile to the profiles of the irreducible components. But recall that the spin parity (see Section 3.3.2) is only defined when the profile contains only odd numbers. Hopefully Lemma 5.6 implies

Corollary 5.7. *Let p be an integer partition which contains only odd terms and π a permutation with profile p. Then the profile of each irreducible component of π contains only odd terms.*

Hence, if π is a permutation with profile p containing only odd numbers, we can discuss about the spin parity of its components. The situation is simpler than the one in Lemma 5.6 as the spin parity does not depend on the structure of the endpoints of each component.

Lemma 5.8. *Let p be a partition with odd parts and π a permutation with profile p. Then the spin parity of π is the sum mod 2 of the spins of the irreducible components of π.*

Proof. Recall that the spin invariant of an irreducible permutation is the Arf invariant of a quadratic form q_π on $\mathbb{F}_2^{\mathcal{A}}$. It is geometrically defined on $H_1(S;\mathbb{Z}/2)$ where $S = S(\pi, \zeta)$ is any suspension of π by

$$q_\pi(x) = (w(x) + \#(\text{components of } x) + \#(\text{self intersection of } x)) \mod 2.$$

In the above formula, $w(\gamma)$ is the winding number of γ which depends on the flat metric of the suspension while the other two are topological. Let π be a permutation and $\pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_k$ its decomposition in irreducible components. Let $S = S(\pi, \zeta)$ be a suspension of π and $S_j = S(\pi_j, \zeta_j)$ the associated suspension of each irreducible components. Then

$$H_1(S;\mathbb{Z}/2) \simeq \bigoplus_{j=1}^{k} H_1(S_j;\mathbb{Z}/2) \quad \text{and} \quad q_\pi = \sum_{j=1}^{k} q_{\pi_j}.$$

To complete the proof, we remark that the Arf invariant is additive (which follows from Theorem 3.13). □

5.2 Removing the ends of a standard permutation

Let π be a standard permutation on the $n+2$ ordered symbols $\{0,1,\ldots,n+1\}$ (i.e. $\pi(0)=n+1$ and $\pi(n+1)=0$). Consider the permutation $\tilde{\pi}$ on the n letters $\{1,\ldots,n\}$ obtained by removing 0 and $n+1$ in π. We call $\tilde{\pi}$ the *degeneration* of π. As a permutation, $\tilde{\pi}$ corresponds to the restriction of the domain of π from $\{0,1,\ldots,n+1\}$ to $\{1,\ldots,n\}$. The term degeneration comes from geometric consideration. Let $(\zeta^{(t)})_{t>0}$ be a continuous sequence of suspensions of π which converges to a vector $\bar{\zeta}\in\mathbb{R}^2$ for which $\bar{\zeta}_0=0=\bar{\zeta}_{n+1}=0$ and $Re(\bar{\zeta}_k)>0$ for all $0<k<n+1$ and the imaginary part of $\bar{\zeta}$ satisfies the condition of suspension for $\tilde{\pi}$. Then the limit $\bar{\zeta}$ is a suspension of $\tilde{\pi}$ which do not live in the same stratum $\Omega\mathcal{M}(p)$ as S but is obtained as a limit of a continuous family $(S_t)\subset\Omega\mathcal{M}(p)$ which degenerates for $t\to\infty$.

The degeneration operation is invertible and gives a bijection between the set of permutations on n letters and standard permutation on $n+2$ letters. We emphasise that the irreducibility property is not preserved. For counting permutations in Rauzy classes, as we did in Section 4 we analyze the geometric surgery associated to this combinatorial operation.

5.2.1 Marked profile, relation between γ^{std} and c

As in Lemma 5.6, the profile of the degeneration depends only on the profile of the initial permutation and its marking. The proposition below expresses the profile of the degeneration from the profile of a standard permutation.

Proposition 5.9. *Let π be a standard permutation. If π has a marked profile of the first type $(m|a,\, p')$, then its degeneration $\tilde{\pi}$ has marked profile $(m-2|m-a-2,\, p')$. If π has a marked profile of the second type $(m_l\odot m_r,\, p')$, then its degeneration $\tilde{\pi}$ has a marked profile $(m_l-1)\odot(m_r-1),\, p')$.*

Proof. We write the standard permutation π, in the following form

$$\pi=\begin{pmatrix} 1\; y_1\; \ldots\; x_0\; 0 \\ 0\; y_0\; \ldots\; x_1\; 1 \end{pmatrix}.$$

Let Γ be the interval diagram of π. If the marking of π is of the first type, let say $m|a$, then the corresponding singluarity in its interval diagram writes $c=(\underline{x_0},\overline{(0,1)},\underline{x_1},A,\overline{y_0},\underline{(0,1)},\overline{y_1},B)$ where A and B are some blocks of lengths respectively $2(m-a-2)$ and $2(a-1)$. Let $\tilde{\pi}=\binom{y_1\ldots x_0}{y_0\ldots x_1}$ be the degeneration of π and $\tilde{\Gamma}$ its interval diagram. The interval diagram $\tilde{\Gamma}$ is obtained from the one of Γ by modifying c as $\tilde{c}=(\underline{(x_0,x_1)},A,\overline{(y_0,y_1)},B)$ where the blocks A and B have not changed. The angle between the left end point and the right endpoint is $|B|$. Hence, the permutation $\tilde{\pi}$ has a marking of the first type $m-2|m-a-2$.

Now, consider the case of a marking of the second type. By symmetry, it is enough to consider one endpoint of the interval. Let c_l be the cycle of the interval diagram that contains the left end point. It writes $c_l=(\underline{x_0},\overline{(0,1)},\underline{x_1},A)$ and becomes $(\underline{(x_0,x_1)},A)$ in the degeneration $\tilde{\pi}$ and proves the the proposition. □

From Proposition 5.9, we deduce a corollary about the relations between the numbers $c(p)$ of Section 4 and the numbers $\gamma^{irr}(p)$ and $\gamma(p)$. For an integer partition p', we denote by $z_{p'}$ the cardinality of the centralizer of any permutation in the conjugacy class associated to p'. Let e_i be the number of parts equal to i in p' then

$$z_{p'} = \prod_{i=1}^{n} i^{e_i} e_i!.$$

Corollary 5.10. *Let $p = (m|a,\, p')$ be a marking of the first type then*

$$\gamma(m|a,\, p') = \gamma^{std}(m+2|m+2-a',\, p') \quad \text{and} \quad \gamma^{std}\,(m|a, p') = \frac{c\left(p_{m|a}\right)}{z_{p'}},$$

Let $p = (m_l \odot m_r,\, p')$ be a marking of the second type then

$$\gamma(m_l \odot m_r,\, p') = \gamma^{std}((m_l+1) \odot (m_r+1),\, p') \quad \text{and} \quad \gamma^{std}\,(m_l \odot m_r, p') = \frac{c\,(p_{m_l \odot m_r})}{z_{p'}}.$$

Proof. The two equalities on the left follows from Proposition 5.9 as the degeneration is a bijection.

Recall that $c(p)$ counts the number of labeled standard permutations while $\gamma^{std}(p)$ counts unlabeled ones. Given a standard permutation π the different ways we have to label it with a fixed labelization $\overline{\sigma_\pi}$ is exactly $z_{p'}$. $\square$

5.2.2 Spin parity, relation between δ^{std} and d

In order to get a counting formula relative to the spin invariant, we now analyze the relation between the spin parity of a standard permutation π and the one of its degeneration.

Proposition 5.11. *Let π be a standard permutation of length $n+2$ and note $\alpha_1 = \pi_t^{-1}(1) = \pi_b^{-1}(n+2)$ and $\alpha_2 = \pi_b^{-1}(1) = \pi_t^{-1}(n+2)$. If π has a marking of the first type $m|a$, then $\tilde{\pi}$ has a spin parity which is $\mathrm{Arf}(q_{\tilde{\pi}}) = \mathrm{Arf}(q_\pi) + a + 1$ modulo 2. If π has marking of the second type, then the spin parity of $\tilde{\pi}$ is the same as the one of the permutation obtained by removing the letter α_1 or α_2 in π.*

Proof. Let π having a marking of the first type and $\tilde{\pi} = \pi_1 \cdot \pi_2 \cdot \ldots \cdot \pi_k$ the decomposition of $\tilde{\pi}$ into irreducible components. We denote by S_π a suspension of π, $S_{\tilde{\pi}}$ a suspension of $\tilde{\pi}$ and S_{π_j} the one induced on each irreducible components. Let ζ_j for $j = 0, \ldots, n+1$ be the sides of the suspension $S_{\tilde{\pi}}$ (see Section 3.3.2 and in particular Figure 3.9). As the marking of π is of type one, both intervals labeled 0 and $n+1$ have the same singularities at both ends. Hence ζ_0 and ζ_{n+1} are elements of $H_1(S;\mathbb{Z}/2)$ and there is a symplectic sum

$$H_1(S;\mathbb{Z}/2) = (\mathbb{Z}/2\;[\zeta_0] \oplus \mathbb{Z}/2\;[\zeta_{n+1}]) \oplus \bigoplus_{j=1}^{k} H_1(S_j;\mathbb{Z}/2).$$

The form q_π diagonalizes with respect to this decomposition as its bilinear form is Ω_π which is the intersection form in $H_1(S;\mathbb{Z}/2)$. We hence only need to compute the restriction of q_π to the symplectic module of rank two $M = (\mathbb{Z}/2\;[\zeta_0] \oplus \mathbb{Z}/2\;[\zeta_{n+1}])$. A direct computation shows that

$$q_\pi(\zeta_0) = w(\zeta_0) + 1 + 0 = a + 1 = q_\pi(\zeta_n) \quad \text{and} \quad q(\zeta_0 + \zeta_1) = 1$$

Hence $Arf(q_\pi|_M) = a+1$. By additivity of the Arf invariant we get $\sum Arf(q_{\pi_j})+a+1 = Arf(q_\pi)$.

Now, we consider a permutation π with marked profile $(m_l \odot m_r, p')$. If $\tilde{\pi}$ has a spin parity then both m_l and m_r are even. If we remove the interval labeled 0 (or $n+1$) the permutation has profile $(m_l + m_r - 1|a, p')$. The conservation of the spin statement is a direct consequence of Lemma 4.8 of the preceding section. $\square$

Let $\delta^{std}(p)$ be the difference between the number of odd spin permutations and even spin permutations among standard permutations with profile p.

Corollary 5.12. *Let $(m|a, p')$ be a marked profil of type one then*

$$\delta(m|a,\, p') = (-1)^{(a+1)}\delta^{std}(m{+}2|m{+}2{-}a',\, p') \quad \textit{and} \quad \delta^{std}\left(m|a, p'\right) = \begin{cases} 0 & \textit{if } a \equiv 0 \mod 2, \\ \frac{d\left(p_{m|a}\right)}{z_{p'}} & \textit{otherwise.} \end{cases}$$

Let $(m_l \odot m_r)$ be a marked profile of type two then

$$\delta(m_l \odot m_r, p') = \frac{d\left(p_{(m_l+1)\odot(m_r+1)}\right)}{z_{p'}} \quad \textit{and} \quad \delta^{std}\left(m_l \odot m_r,\, p'\right) = \frac{d\left(p_{m_l \odot m_r}\right)}{z_{p'}}$$

Proof. The proof is similar to the one of Corollary 5.10. The left equalities follows from Proposition 5.11 and the right ones from the definition of d. $\square$

5.3 Counting permutations in Rauzy classes

5.3.1 Marked points and hyperelliptic strata

As we did in Section 4.3.1 with labeled standard permutations, we give a relation between cardinalities of a Rauzy diagram and the ones obtained by adding marked points. As a corollary, we get the cardinality of any hyperelliptic Rauzy class.

Let $\mathbf{p}$ be a marked profile which corresponds to an hyperelliptic strata $\Omega\mathcal{M}(2g-1, 1^k)$ or $\Omega\mathcal{M}(g, g, 1^k)$. We denote by $hyp(\mathbf{p})$ the number of irreducible permutations with marked profile p. From the explicit description of the Rauzy class associated to rotation class permutation and hyperelliptic class (Section 3.1.3) we get the two following proposition.

Proposition 5.13. *We have*

$$\gamma^{irr}(1|0,\, (1^{n-2})) = n-1 \quad \textit{and} \quad \gamma^{irr}(1 \odot 1,\, (1^{n-3})) = \frac{(n-1)(n-2)}{2}.$$

If n is even the profile of π is $p = (n-1)$ and the genus of a suspension of π is $g = n/2$. In this case for $a \leq g-1$ we have

$$hyp(2g-1|a) = hyp(2g-1|2g-1-a) = \binom{2g-1}{2a+1}$$

If n is odd, the profile of π is $((n-1)/2, (n-1)/2)$ and the genus of a suspension is $g = (n-1)/2$. In this case for $a \leq g$ we have

$$hyp(2g-1|a) = hyp(2g-1|2g-a) = \binom{2g-1}{2a+1} \quad \textit{and} \quad hyp(g \odot g) = \sum_{k=0}^{g-1}\binom{2g}{2k} = 2^{2g-1}-1$$

Let $\mathcal{C} \subset \Omega\mathcal{M}(\kappa)$ be a connected component of a stratum and $\mathcal{R}$ its associated Rauzy diagram. We assume that the partition κ does not contain 1. Consider $\mathcal{C}_0 \subset \Omega\mathcal{M}(\kappa \uplus 0^k)$ the connected component obtained by marking k points in the surfaces of $\mathcal{C}$. Let $\mathcal{R}_0$ be the extended Rauzy diagram associated to $\mathcal{R}_0$. The following theorem shows that the cardinality of $\mathcal{R}_0$ is a linear combination of the cardinality of $\mathcal{R}$ and the number of standard permutations in $\mathcal{R}$. Recall that $\mathcal{R}(m)$ denotes for $m-1$ an element of κ the Rauzy class which correspond to the elements $\pi \in \mathcal{R}$ such that the left end point has an angle $2m\pi$ (see Section 3.4.1).

Theorem 5.14. *Let $\mathcal{R}$, $\mathcal{R}_0$ and k be as above. Le let d be the number of letters in the permutations of $\mathcal{R}$, r the number of standard permutations in $\mathcal{R}$ and m an element of the profile of $\mathcal{R}$. Then*

$$|\mathcal{R}_0(m)| = \binom{d+k}{k} |\mathcal{R}(m)| \quad \textit{and} \quad |\mathcal{R}_0(1)| = \binom{d+k}{k-1} |\mathcal{R}| + \binom{d+k}{k-1}\, d\, r$$

In particular, for the cardinalities of extended Rauzy classes, we get the following relations

$$|\mathcal{R}_0| = \binom{d+k+1}{k}\ |\mathcal{R}| + \binom{d+k}{k-1}\ d\, r.$$

The proof of the theorem follows from Proposition 5.16 below. As a corollary of the theorem, we get an explicit formula for the cardinality of Rauzy diagrams associated to any hyperelliptic component of stratum.

Corollary 5.15. *Let $\mathcal{R}$ be the extended Rauzy diagram of the hyperelliptic component $\Omega\mathcal{M}_{hyp}(2g-1, 1^k)$ (reps. $\Omega\mathcal{M}(g, g, 1^k)$) for which $d = 2g$ (resp. $d = 2g+1$) is the number of intervals in $\Omega\mathcal{M}(2g-1)$ (resp. $\Omega\mathcal{M}(g, g)$). Then the cardinality of the Rauzy diagrams are*

$$|\mathcal{R}(1)| = \binom{d+k}{k-1} (2^{d-1} + d - 1) \quad \textit{and} \quad |\mathcal{R}(d-1)| = \binom{d+k}{k} (2^{d-1} - 1).$$

The cardinality of the extended Rauzy diagram is

$$|\mathcal{R}| = \binom{d+k+1}{k} (2^{d-1} - 1) + \binom{d+k}{k-1} d.$$

We now prove Theorem 5.14. As above, let $\mathcal{R}$ be an extended Rauzy class and $\mathcal{R}_0$ the one obtained by adding k marked points. We denote by p the profile of the permutations in $\mathcal{R}$ and we assume that $1 \notin p$. If $m|a$ (resp. $m_l \odot m_r$) is a marking of the first type (resp. second type) then we denote by $\mathcal{R}(m|a)$ (resp. $\mathcal{R}(m_l \odot m_r)$) the elements of the extended Rauzy class $\mathcal{R}$ which has marking $m|a$ (resp. $m_l \odot m_r$).

Proposition 5.16. *Let $\mathcal{R}$ and $\mathcal{R}_0$ be extended Rauzy classes as above, then*

1. $|\mathcal{R}_0(m|a)| = \binom{d+k-1}{k} |\mathcal{R}(m|a)|$,
2. $|\mathcal{R}_0(m_l \odot m_r)| = \binom{d+k-1}{k} |\mathcal{R}(m_l \odot m_r)|$,
3. $|\mathcal{R}_0(m \odot 1)| = |\mathcal{R}_0(1 \odot m)| = \binom{d+k-1}{k-1} |\mathcal{R}(m)|$,
4. $|\mathcal{R}_0(1 \odot 1)| = \binom{d+k-1}{k-2} (|\mathcal{R}| + d\, r)$,

5. $|\mathcal{R}_0(1|0)| = \binom{d+k-1}{k-1}\, d\, r$.

Proof. We first prove equalities *1* and *2*. Let $\pi \in \mathcal{R}$ with marking $m|a$ or $m_l \odot m_r$ and $P_0 \subset \mathcal{R}_0$ the set of permutations π_0 with the same marking as π and such that they are obtained from π by adding k zeroes. The marked points of any $\pi_0 \in P_0$ belong inside the d intervals. Hence $|P_0| = \binom{d+k-1}{k}$ is the number of choices of placing k undifferentiated points in d intervals.

Now, we proove equality *3*. Let $\pi \in \mathcal{R}(m)$ and $P_0 \subset \mathcal{R}_0(m\odot 1)$ the set of permutations obtained from π by adding k marked points. For any $\pi_0 \in P_0$, because of the marking $m\odot 1$ and $1 \notin p$ one of the marked point has to go to the right endpoint of the permutation. There is only one way to do this by the following operation

$$\pi = \begin{pmatrix} \ldots\ y\ \ldots\ x \\ \ldots\ x\ \ldots\ y \end{pmatrix} \mapsto \pi_0 = \begin{pmatrix} \ldots\ y\ c\ \ldots\ x \\ \ldots\ x\ c\ \ldots\ y \end{pmatrix}.$$

Then, the $k-1$ other marked points belong in the $d+1$ intervals and the number of choices for such operation is $\binom{(d+1)+(k-1)-1}{k-1} = \binom{d+k-1}{k-1}$. Hence $|P_0| = \binom{d+k-1}{k-1}$.

Equality *4* is similar to equality *3* but two of the marked points have to be placed at the extremities. We get the coefficient $\binom{(d+2)+(k-2)-1}{k-2} = \binom{d+k-1}{k-2}$.

We now proove equality *5*. Let $\pi \in \mathcal{R}_0$ be a permutation with marking $1|0$. Then we can write a general form for π_0 and we see below that removing the marked point of π gives a standard permutation.

$$\pi_0 = \begin{pmatrix} a_0 & A & b_1 & a_1 & B & b_0 \\ a_1 & C & b_0 & a_0 & D & b_1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} a & A & c & B & b \\ b & C & c & D & a \end{pmatrix}. \tag{5.4}$$

Hence, the only way to mark 1 point on a permutation in $\mathcal{R}$ in order to obtain a marking $1|0$ is that π is standard. Starting from a standard permutation $\pi \in \mathcal{R}$ the construction of a permutation π_0 with marking $1|0$ is as follows. Choose the letter c which will play the role of an intermediate and place it as in (5.4). There are d choices for the letter c. Then, the other $k-1$ points can be placed inside $d+1$ intervals. We get exactly $\binom{d+k-1}{k-1}\, d$ permutations in $\mathcal{R}_0$ built from π. □

5.3.2 The number of irreducible permutations

Before counting permutations in Rauzy diagrams, we recall the elementary method to count irreducible permutations. Most of the idea developed here are similar to the one we will use in the next section. As in (5.1), let $f(n) = |S_n^o|$ be the number of irreducible permutations of length n. We recall the elementary method for different expression of (5.1) and get an asymptotic development. See the original article [Com72] for further details on asymptotics and [FS09] for general considerations about the relations between generating series and operations on combinatorial classes.

Let $E(t) = \sum n!\ t^n$ and $F(t) = \sum f(n)\ t^n$ considered as formal serie. Given a permutation, its factorization in irreducible elements is unique. In terms of the generating functions E and F, the equation (5.1) can be seen as

$$E = \frac{1}{1-F} = \sum_{k=0}^{\infty} F^k.$$

Using an inclusion/exclusion argument, we get a dual formulation of the equation (5.1)

$$f(n) = \sum_{k=1}^{n} \sum_{c_1+\ldots+c_k=n} (-1)^{k+1}\, c_1! \,\ldots\, c_k! \qquad \text{or} \qquad F = 1 - \frac{1}{E} = 1 - \sum_{k=0}^{\infty}(1-E)^k. \quad (5.5)$$

We can write a simpler relation between factorial numbers $n!$ and the numbers $f(n)$. Any permutation can be decomposed uniquely as the product of an irreducible permutation and a permutation. Hence

$$\sum_{i=1}^{n} f(i)\,(n-i)! = n! \qquad \text{or} \qquad EF = E - 1. \quad (5.6)$$

From the equations on generating functions, we see that the formulas (5.1), (5.5), (5.6) are equivalent. However each one has its own advantage: equation (5.1) is the most natural, equation (5.5) gives a closed formula and (5.6) is adapted for explicit computations.

The equation (5.6) suffices to obtain an equivalent of the number of irreducible permutations. For an asymptotic serie, see [Com72].

Proposition 5.17 ([Com72])**.** *$f(n)$ is equivalent to $n!$ (e.g. $f(n) = n!(1+o(1))$).*

Proof. Let $g(n) := f(n)/n!$. Those numbers satisfy the inequality $g(n) \leq 1$ and from Equation (5.6) we get

$$\begin{aligned} g(n) &= 1 - \sum_{k=1}^{n-1} g(k) \binom{n}{k}^{-1} \\ &= 1 - \frac{2}{n} - \sum_{k=2}^{n-2} \binom{n}{k}^{-1} \\ &\geq 1 - \frac{2}{n} - (n-3)\,\frac{2}{n(n-1)}. \end{aligned}$$

As the right member of this equation tends to 1 we get that $g(n)$ tends to 1 as n tends to ∞. □

5.3.3 Formula for γ^{irr} and δ^{irr}, proof of Theorem 2.3

Recall that $\gamma(m|a,\, p')$ and $\gamma(m_l \odot m_r,\, p')$ (resp. $\gamma^{irr}(m|a,\, p')$ and $\gamma^{irr}(m_1 \odot m_2,\, p')$) denote the number of permutations (resp. irreducible permutations) with marked profile $(m|a,\, p')$ and $(m_1 \odot m_2,\, p')$. The numbers $\gamma(m|a,\, p')$ and $\gamma(m_l \odot m_r,\, p')$ are related to the number $c(p)$ of Section 4 by Corollary 5.10.

The two formulas in Theorem 2.3 are obtained by an exclusion procedure and are very similar to (5.6) which gives an explicit formula for the number $f(n)$ of irreducible permutations in S_n in the following form

$$f(n) = n! - \sum_{k=1}^{n-1} f(k)(n-k)!.$$

In the above formula, $n!$ corresponds to the cardinality of permutations and the summation corresponds to all reducible ones. Each reducible permutation has to be thought as the concatenation of an irreducible permutation of length k with any permutation of length $n-k$.

We explain the formula for $\gamma^{irr}(m_l \odot m_r, p')$, the other being similar. The set of all permutations (non necessarily reducible) with marked profile $m_l \odot m_r, p')$ is exactly $\gamma^{std}((m_l+1) \odot (m_r+1), p')$ (see Section 5.2 and in particular Proposition 5.9). Then we have to subtract all irreducible. Recall from Lemma 5.6 that the profile of a reducible permutation can be expressed in terms of its irreducible components. We consider the possible form of a reducible permutation $\pi_1\pi_2$ with marked profile $(m_l \odot m_r, p')$ where π_1 is irreducible.

1. either π_1 has a marking of the first type and π_2 a marking of the second type,
2. or π_1 has marking of the second type and π_2 of the first type,
3. or π_1 and π_2 both have marking of the second types.

The three cases above, correspond to the three summations in the formula $\gamma^{irr}(m_l \odot m_r, p')$ in Theorem 2.3.

5.3.4 Explicit formula for profile $(2g-1)$

We gave in Section 5.3.1 examples of family of Rauzy classes obtained by adding marked points. Theorem 5.14 gives an explicit formula for the behavior of the cardinality. In those example, the genus was fixed. In this section we consider the family of Rauzy diagrams which are the Rauzy classes associated to the odd and even components of $\Omega\mathcal{M}(2g-1)$. This family of strata are the so called *minimal strata*. Recall that for $g=2$, $\Omega\mathcal{M}(2g-1)$ has only one connected components, for $g=3$ there are 2 and for $g \geq 4$ there are three. The cardinality of the hyperelliptic component is given in Proposition 3.6. To get the cardinality of all Rauzy classes, we consider explicit formulas for the numbers $\gamma^{irr}(2g-1)$ and $\delta^{irr}(2g-1)$ in the following proposition.

Proposition 5.18. *Let $n = 2g-1$ then we have the following formulas for $\gamma^{irr}((n))$ and $\delta^{irr}((n))$*

$$\gamma^{std}((n)) = \frac{(n-1)!}{n+1} \quad \textit{and} \quad \delta^{std}((n)) = \frac{(n-1)!}{2^{n-1}}\ , \tag{5.7}$$

$$\gamma^{irr}((n)) = \sum_{k=1}^{2n+1} (-1)^{k+1} \sum_{\substack{(c_1,\ldots,c_k)\\ \sum c_j = n+1}} \prod_{i=1}^{k} \frac{(2\ c_i)!}{c_i+1}\ , \tag{5.8}$$

$$\delta^{irr}((n)) = \frac{1}{2^{n+1}} \sum_{k=1}^{2n+1} (-1)^{k+1} \sum_{\substack{(c_1,\ldots,c_k)\\ \sum c_j = n+1}} \prod_{i=1}^{k} (2c_i)!\ . \tag{5.9}$$

Proof. This is a direct consequence of Corollary 4.14 and the explicit formula for d in 4.19. □

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